A FIRST COURSE IN ALGEBRA

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BOSTON, U.S.A.

D. C. HEATH & CO., PUBLISHERS

1909

PREFACE

In the preparation of this text the author acknowledges joint-authorship with Robert L. Short.

This book meets the demand that the pupil be given an elementary algebra containing no more than can be accomplished in the time allotted to the subject. It is not intended for a complete course, but gives the student a good working knowledge of the subject through simultaneous quadratics. It should be followed by a second course by those intending to pursue the study of higher mathematical subjects. This book is sufficient preparation for geometry, and the frequent introduction of geometric ideas and geometric problems not only prepares for geometry but also makes that subject attractive to the learner.

This text is as brief as the algebra of years ago, and yet contains all that is good in modern mathematical thought. Attention is called to the introduction of graphical methods through simple horizontal and vertical measurements (Exercise 4, Exercise 41, problems 28–30). This procedure makes the transition to Cartesian coördinates a natural one. Teachers will find that the color scheme recommended in graphs will greatly aid the student in connecting related data. Pedagogical advantage is gained through the combining of related and reverse processes. (Chapters III, VII, X, XII, XIII.)

The use of the fractional exponent in operations involving surds is recommended, thereby avoiding confusion, since the four fundamental laws and the exponential laws of Multiplication, Division, Involution, and Evolution, are the only ones involved. The complete index will be found helpful to both pupil and teacher. No attempt is made toward technical

definition. Definitions for the beginner must be explanatory and descriptive. The lists of queries will aid in fixing both definitions and principles.

The authors thank the many teachers of mathematics who have made this book better and have brought it close to actual class-room conditions by their timely criticism and suggestion.

WEBSTER WELLS.

CONTENTS

ALGEBRA

I. DEFINITIONS AND NOTATION

SYMBOLS REPRESENTING NUMBERS

1. In **Algebra** the symbols usually employed to represent numbers are the *Arabic numerals* and the *letters of the alphabet.*

The numerals represent known numbers.

The letters represent numbers which may have any values whatever, or numbers whose values are to be found.

EQUATIONS

2. The **Sign of Equality**, =, is read "*equals.*"

Thus, $a=b$ signifies that the number a equals the number b.

3. An **Equation** is an expression of equality.

The *first member* of an equation is the number to the left of the sign of equality, and the *second member* is the number to the right of that sign; thus, in the equation $2x-3=5$, the first member is $2x-3$, and the second member 5.

AXIOMS

4. An **Axiom** is a statement which is assumed as self-evident. Algebraic operations of finite numbers are based in part on the following axioms:

1. Any number equals itself.
2. Any number equals the sum of all its parts.
3. Any number is greater than any of its parts.
4. Two numbers which are equal to the same number, or to equal numbers, are equal.

5. If the same number, or equal numbers, be added to equal numbers, the resulting numbers will be equal.

6. If the same number, or equal numbers, be subtracted from equal numbers, the resulting numbers will be equal.

7. If equal numbers be multiplied by the same number, or equal numbers, the resulting numbers will be equal.

8. If equal numbers be divided by the same number, or equal numbers, the resulting numbers will be equal. Numbers cannot be divided by the number 0.

SOLUTION OF PROBLEMS BY ALGEBRAIC METHODS

5. The following examples illustrate some uses of algebraic symbols:

1. The sum of two numbers is 30, and the greater exceeds the less by 4; what are the numbers?

We will represent the less number by x.

Then the greater will be represented by $x+4$.

By the conditions of the problem, the sum of the *less* number and the *greater* is 30; this is stated in algebraic language as follows:

$$x+x+4=30. \qquad (1)$$

$$x+x=2x.$$

Therefore, $$2x+4=30.$$

The members of this equation, $2x+4$ and 30, are equal numbers; if from each of them we subtract the number 4, the resulting numbers will be equal (Ax. 6, § 4).

Therefore, $$2x=30-4, \text{ or } 2x=26.$$

Dividing the equal numbers $2x$ and 26 by 2 (Ax. 8, § 4), we have

$$x=13.$$

Hence, the less number is 13, and the greater is $13+4$, or 17.

The written work will stand as follows:

Let	$x=$ the less number.
Then,	$x+4=$ the greater number.
By the conditions,	$x+x+4=30$, or $2x+4=30$.
Whence,	$2x=26$.
Dividing by 2,	$x=13$, the less number.
Whence,	$x+4=17$, the greater number.

2. The sum of the ages of A and B is 109 years, and A is 13 years younger than B; find their ages.

Let n represent the number of years in B's age.

Then, $n-13$ will represent the number of years in A's age.

By the conditions of the problem, the sum of the ages of A and B is 109 years.

Whence, $n-13+n=109$, or $2n-13=109$.

Adding 13 to both members (Ax. 5, § 4),

$$2n=122.$$

Dividing by 2, $n=61$, the number of years in B's age.

And, $n-13=48$, the number of years in A's age.

The written work will stand as follows:

Let $n=$ the number of years in B's age.

Then, $n-13=$ the number of years in A's age.

By the conditions, $n-13+n=109$, or $2n-13=109$.

Whence, $2n=122$.

Dividing by 2, $n=61$, the number of years in B's age.

Therefore, $n-13=48$, the number of years in A's age.

In Ex. 2, we do not say "let n represent B's *age*," but "let n represent the *number of years* in B's age."

3. A, B, and C together earn $66. A's share is one-half as much as B's, and C's is 3 times as much as A's. How much has each?

Let $x=$ the number of dollars A has.

Then, $2x=$ the number of dollars B has,

and $3x=$ the number of dollars C has.

By the conditions, $x+2x+3x=66$.

But the sum of x, twice x, and 3 times x is 6 times x, or $6x$.

Whence, $6x=66$.

Dividing by 6, $x=11$, the number of dollars A has.

Whence, $2x=22$, the number of dollars B has,

and $3x=33$, the number of dollars C has.

(By letting x represent the number of dollars A has, in Ex. 3, we avoid fractions.)

EXERCISE 1

Write the following in algebraic symbols:

1. One number is 4 more than another. What is their sum? (Hint: Let $x =$ the smaller number.)

2. There are three numbers such that the second is twice the first, and the third thrice the first. What is their sum?

3. The sum of two numbers is 20 and one of the numbers is x. What is the other number?

4. If one number is 4 times another, what is their difference?

5. Write: the sum of 5 times a certain number and 3 times the number, divided by 3.

6. The sum of two numbers is a and one of the numbers is b. What is the other?

7. The greater of two numbers is 8 times the less, and exceeds it by 49; find the numbers.

8. The sum of the ages of A and B is 119 years, and A is 17 years older than B; find their ages.

9. Divide $74 between A and B so that A may receive $48 more than B.

10. Divide $108 between A and B so that A may receive 5 times as much as B.

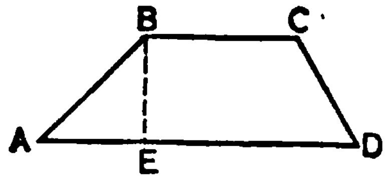
11. Divide 91 into two parts such that the smaller shall be one-sixth of the greater.

12. A man travels 112 miles by train and steamer; he goes by train 54 miles farther than by steamer. How many miles does he travel in each way?

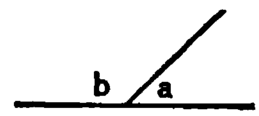
13. The sum of three numbers is 69; the first is 14 greater than the second, and 28 greater than the third. Find the numbers.

14. The area of a trapezoid is equal to the product of one-half the sum of the parallel sides and the altitude. In the trapezoid $ABCD$, AD is 8 more than BC, EB is 6, and the

area of the trapezoid is 54. Find the length of BC and of AD. Is the drawing correct?



15. The sum of two angles a and b is 180°. The angle b is three times as great as the angle a. Find the number of degrees in each angle.

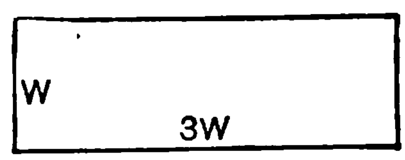


16. Divide \$6.75 between A and B so that A may receive one-fourth as much as B.

17. A man has \$2. After losing a certain sum, he finds that he has left 20 cents more than 3 times the sum which he lost. How much did he lose?

18. A, B, and C in partnership gain \$140; A is to have 4 times as much as B, and C as much as A and B together. Find the share of each.

19. One side of a rectangle is thrice the side adjacent to it. The opposite sides are equal, and the sum of the sides is 24 inches; find the length and breadth.



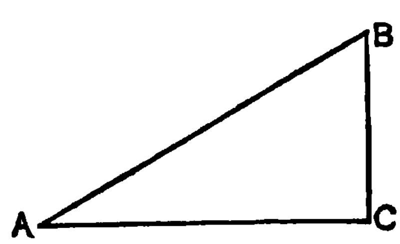
20. At an election two candidates, A and B, had together 653 votes, and A was beaten by 395 votes. How many did each receive?

21. A field is 7 times as long as it is wide, and the distance around it is 240 feet. Find its dimensions.

22. My horse, carriage, and harness are worth together \$325. The horse is worth 6 times as much as the harness, and the carriage is worth \$65 more than the horse. How much is each worth?

23. The sum of three numbers is 87; the third number is one-eighth of the first, and the second number 15 less than the first. Find the numbers.

24. The sum of the three angles of a triangle is always 180°. In a triangle ABC, angle B is 30° larger than angle A, and angle C is 30° larger than angle B. Find the number of degrees in each angle.



25. The sum of the ages of A, B, and C is 110 years; B's age exceeds twice C's by 12 years, and A is 9 years younger than B. Find their ages.

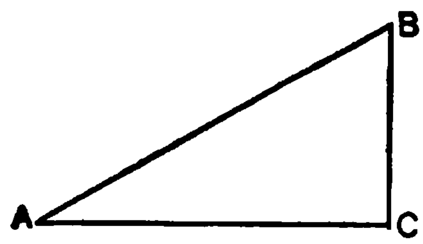
26. A pole 77 feet long is painted red, white, and black; the red is one-fifth of the white, and the black 21 feet more than the red. How many feet are there of each color?

27. Divide 70 into three parts such that the third part shall be one-fifth of the first, and one-fourth of the second.

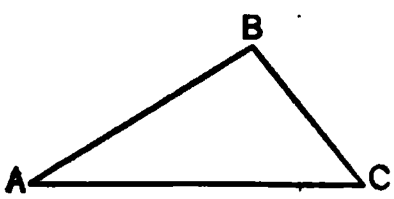
28. In an algebra class of 27 pupils there are twice as many girls as boys. How many girls in the class?

29. A, B, and C have together \$22.50; B has \$1.50 more than A, and C has \$8 less than twice the amount that A has. How much has each?

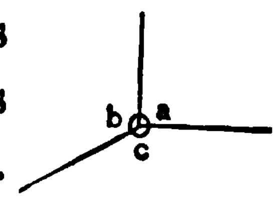
30. In a triangle, ABC, angle C is 90°, angle B is twice angle A. The sum of the three angles is 180°. Find the angles A and B.



31. The sum of the three sides of a triangle, ABC, is 35 feet. Side AB is 4 feet more than side BC and side AC is 7 feet more than side BC. Find the length of AB and AC.



32. Three straight lines are drawn from a point O forming the angles a, b, and c. b is 30° larger than a; c is 30° larger than b. The sum of the three angles is 360°. Find the number of degrees in each angle.



DEFINITIONS

6. The continued product of a number by itself any number of times is called a **Power** of that number.

An **Exponent** is a number written at the right of, and above another number called the **Base**, to indicate what power of the latter is to be taken; thus,

a^2, read "*a square*," or "*a second power*," denotes $a \times a$;
a^3, read "*a cube*," or "*a third power*," denotes $a \times a \times a$;
a^4, read "*a fourth*," "*a fourth power*," or "*a exponent* 4," denotes $a \times a \times a \times a$, etc.

The meaning of exponent will be extended in Chap. XIII. If no exponent is expressed, the *first* power is understood. Thus, a is the same as a^1.

7. Symbols of Aggregation.

The *parentheses* ( ), the *brackets* [ ], the *braces* { }, and the *vinculum* ——, indicate that the numbers enclosed by them are to be taken collectively; thus,

$$(a+b) \times c,\ [a+b] \times c,\ \{a+b\} \times c, \text{ and } \overline{a+b} \times c$$

all indicate that the result obtained by adding b to a is to be multiplied by c.

If an expression involves *parentheses*, the operations indicated *within* the parentheses must be performed first.

EXERCISE 2

Write the following in symbols:

1. The result of subtracting 6 times n from 5 times m.

2. Three times the product of the eighth power of m and the ninth power of n.

3. The quotient of the sum of a and b divided by the sum of c and d.

What operations are signified by the following?

4. $2x^5y^6$.

5. $m(x-y)$.

6. $\dfrac{cx}{mh}$.

7. $3+(y-z)$.

8. $3-(y+z)$.

9. $(m-n)^4$.

10. $\dfrac{a}{b}-\dfrac{c}{d}$.

11. $\left(\dfrac{x+z}{x-z}\right)^2$.

12. $(2a+b)(4c-5d)$.

13. $\left(\dfrac{1}{x}+\dfrac{1}{y}\right)z^4$.

Write the following in symbols:

14. The product of $3x+y$ and z^2.

15. The result of subtracting $y-z$ from x.

16. The product of $a-b$ and $c-d$.

17. The result of adding the quotient of m by n, and the quotient of x by y.

18. The square of $m+n$.

19. The cube of $a-b+c$.

20. Translate into English $\frac{a+b}{2}$; $\frac{a-b}{2}$.

21. In the above example is a a number? What value has it? If a were 5 and b were 3, what would be the value of the fraction?

22. Translate into English $\frac{x+y}{x-y}$.

23. In example 22, if x is 7 and y is 5, find the fraction.

ALGEBRAIC EXPRESSIONS

8. An **Algebraic Expression**, or simply an **Expression**, is a number expressed in algebraic symbols; as,

$$2, \quad a, \quad \text{or } 2x^2-3ab+5.$$

9. The **Numerical Value** of an expression is the result obtained by substituting particular numerical values for the letters involved in it, and performing the operations indicated.

1. Find the numerical value of the expression

$$4a+\frac{6c}{b}-d^3,$$

when $a=4$, $b=3$, $c=5$, and $d=2$.

We have, $4a+\frac{6c}{b}-d^3=4\times4+\frac{6\times5}{3}-2^3=16+10-8=18.$

2. Find the numerical value, when $a=9$, $b=7$, and $c=4$, of

$$(a-b)(b+c)-\frac{a+b}{b-c}.$$

First perform the operations indicated in parentheses.
We have, $a-b=2$, $b+c=11$, $a+b=16$, and $b-c=3$.
Then the numerical value of the expression is

$$2\times 11-\frac{16}{3}=22-\frac{16}{3}=\frac{50}{3}.$$

EXERCISE 3

Find the numerical values of the following when $a=6$, $b=3$, $c=4$, $d=5$, $m=3$, and $n=2$:

1. a^2b-cd^2.
2. $2\,abcd$.
3. $3\,ab+4\,bc-5\,cd$.
4. a^mb^n.
5. a^b+b^a.
6. $\dfrac{a}{bc}+\dfrac{c}{ad}$.
7. $\dfrac{1}{b}+\dfrac{1}{c}-\dfrac{1}{d}$.
8. $\dfrac{15\,c^m}{28\,d^n}$.
9. $\dfrac{a}{b}-\dfrac{b}{c}+\dfrac{c}{d}$.
10. $\dfrac{1}{a}+\dfrac{1}{b}-\dfrac{1}{c}-\dfrac{1}{d}$.
11. $\dfrac{5\,c^2d}{a^3}-\dfrac{cd^2}{2\,b^3}$.
12. $\dfrac{b^2}{a^2}+\dfrac{c^2}{b^2}-\dfrac{d^2}{c^2}$.

Find the numerical values of the following when $a=5$, $b=2$, $c=3$, and $d=4$:

13. $\left(\dfrac{2\,a+d}{2\,b+c}\right)^2$.
14. $(a^2-b^2-d^2)^3$.
15. $5\,a^2(a-b)-2\,b^3(c+d)$.
16. $8(a-b)^2+3(c+d)^2$.
17. $(a-b)^2+(2\,a-3\,b)^2-(b+c)^2$.
18. $(2\,a-b-c+d)(2\,a+b+c-d)$.
19. $\dfrac{8\,a+3\,b-6\,c}{9\,a-4\,b-3\,c}$.
20. $\dfrac{a-b}{a+b}+\dfrac{a-c}{a+c}+\dfrac{a-d}{a+d}$.

Find the numerical values of the following when $a=\frac{3}{4}$, $b=\frac{5}{2}$, $c=\frac{1}{3}$, and $x=4$:

21. $\dfrac{a+c}{a-c}-\dfrac{a-c}{a+c}$.
22. $\dfrac{8\,a+6\,b-15\,c}{16\,a+10\,b+9\,c}$.
23. $x^3+(2\,a+3\,b)x^2-(5\,a-4\,c)x+\frac{8}{3}\,abc$.
24. $x^3-\dfrac{2}{a}+\dfrac{3}{b}x^2+\dfrac{6}{a}-\dfrac{5}{b}+\dfrac{4}{c}x-\dfrac{13}{abc}$.

II. POSITIVE AND NEGATIVE NUMBERS

10. In financial transactions, we may have *assets* or *liabilities*, and *gains* or *losses*; we may have motion along a straight line in a certain direction, or in the *opposite* direction; etc. Taken in pairs, these ideas have opposite meanings or opposite sense.

In each of these cases, the effect of combining with a magnitude of a certain kind another of the opposite kind is to diminish the former, destroy it, or reverse its state.

Thus, if to a certain amount of asset we add a certain amount of liability, the asset is diminished, destroyed, or changed into liability.

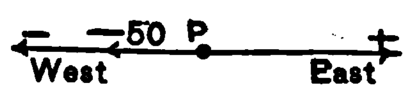
11. The signs + and −, besides denoting addition and subtraction, are also used, in Algebra, to distinguish between the opposite states of magnitudes like those of § 10.

Thus, we may indicate *assets* by the sign +, and *liabilities* by the sign −; for example, the statement that a man's possessions are −\$100 means that he has liabilities to the amount of \$100.

EXERCISE 4

1. If a man has assets of \$400, and liabilities of \$600, how much is he worth?

2. If gains be taken as positive, and losses as negative, what does −\$100 mean?

3. In what position is a man who is −50 feet east of a certain point P? See figure. 

4. In what position is a man who is −3 miles north of a certain place? Draw the figure showing this.

5. How many miles north of a certain place is a man who goes 5 miles north, and then 9 miles south? Draw figure.

12. Positive and Negative Numbers.

If the positive and negative states of any concrete magnitude be expressed *without reference to the unit*, the results are called *positive* and *negative numbers*, respectively.

MULTIPLICATION OF POSITIVE AND NEGATIVE NUMBERS

17. If one algebraic expression is multiplied by another, the first is called the **Multiplicand**, and the second the **Multiplier.**

18. We shall retain for multiplication, in Algebra, its arithmetical meaning, *so long as the multiplier is a positive integer or a positive fraction.* That is, to multiply a number by a positive integer is to *add* the multiplicand as many times as there are units in the multiplier.

For example, to multiply -4 by 3, we add -4 three times.

Thus, $(-4)\times(+3)=(-4)+(-4)+(-4)=-12.$

19. In Arithmetic, the product of two numbers is the same in whichever *order* they are multiplied, that is, whichever is taken as the multiplier.

Thus, 3×4 and 4×3 are each equal to 12.

If we could assume this law to hold for the product of a positive number by a negative, we should have

$$(+3)\times(-4)=(-4)\times(+3)=-12\ (\S\ 18)=-(3\times 4).$$

Then, if the above law is to hold, we must give the following meaning to multiplication by a negative number:

To multiply a number by a negative number is to multiply it by the absolute value (§ 13) of the multiplier, and change the sign of the result.

Thus, to multiply $+4$ by -3, we multiply $+4$ by $+3$, giving $+12$, and change the sign of the result.

That is, $(+4)\times(-3)=-12.$

Again, to multiply -4 by -3, we multiply -4 by $+3$, giving -12 (§ 18), and change the sign of the result.

That is, $(-4)\times(-3)=+12.$

20. From §§ 18 and 19 we derive the following rule:

To multiply one number by another, multiply together their absolute values.

Make the product *plus* when the multiplicand and multiplier are of *like* sign, and *minus* when they are of *unlike* sign.

21. Examples.

1. Multiply $+8$ by -5.

By the rule, $(+8)\times(-5) = -(8\times 5) = -40.$

2. Multiply -7 by -9.

By the rule, $(-7)\times(-9) = +(7\times 9) = +63.$

3. Find the numerical value when $a=4$ and $b=-7$, of

$$(a+b)^3.$$

We have,
$$(a+b)^3 = (4-7)(4-7)(4-7)$$
$$= (-3)(-3)(-3) = -27.$$

EXERCISE 6

Find the values of the following:

1. $(+5)\times(-4)$.
2. $(-11)\times(+3)$.
3. $(-8)\times(-7)$.
4. $(+9)\times(-6)$.
5. $(-12)\times(+9)$.
6. $(-24)\times(-5)$.
7. $(-14)\times(+15)$.
8. $(+27)\times(-19)$.
9. $\left(-\frac{7}{8}\right)\times\left(-\frac{9}{5}\right)$.

III. ADDITION AND SUBTRACTION OF ALGEBRAIC EXPRESSIONS. PARENTHESES

22. A **Monomial, or Term,** is an expression (§ 8) whose parts are not separated by the sign $+$ or $-$; as $2x^2, -3ab$, or 5.

$2x^2$, $-3ab$, and $+5$ are called the *terms* of the expression $2x^2-3ab+5$.

A **Positive Term** is one preceded by a $+$ sign; as $+5a$. If no sign is expressed, the term is understood to be positive.

A **Negative Term** is one preceded by a $-$ sign; as $-3ab$. The $-$ sign must never be omitted before a negative term.

23. If two or more numbers are multiplied together, each of them, or the product of any number of them, is called a **Factor** of the product.

Thus, a, b, c, ab, ac, and bc are factors of the product abc.

24. Any factor of a product is called the **Coefficient** of the product of the remaining factors.

Thus, in $2ab$, 2 is the coefficient of ab, $2a$ of b, a of $2b$, etc.

25. If one factor of a product is expressed in *Arabic numerals*, and the other in *letters*, the former is called the *numerical coefficient* of the latter.

Thus, in $2ab$, 2 is the numerical coefficient of ab.

If no numerical coefficient is expressed, the coefficient 1 is understood; thus, a is the same as $1a$.

26. By § 20, $(-3)\times a = -(3\times a) = -3a$.

That is, $-3a$ is the product of -3 and a.

Then, -3 is the *numerical coefficient* of a in $-3a$.

Thus, *in a negative term as in a positive, the numerical coefficient includes the sign.*

27. **Similar** or **Like Terms** are those which either do not differ at all, or differ only in their numerical coefficients; as $2x^2y$ and $-7x^2y$.

Dissimilar or **Unlike Terms** are those which are not similar; as $3x^2y$ and $3xy^2$.

ADDITION OF MONOMIALS

28. The result of addition is called the **Sum.**

29. The adding of b to a is expressed $a+b$. The sum is $(a+b)$. But where no ambiguity is to be feared, parentheses may be omitted.

30. The addition of monomials is effected by uniting them with their respective signs.

Thus, the sum of a, $-b$, c, $-d$, and $-e$ is

$$a-b+c-d-e.$$

31. We assume that the terms can be united *in any order*, provided each has its proper sign.

Hence, the result of § 30 can also be expressed

$$c+a-e-d-b,\ -d-b+c-e+a, \text{ etc.}$$

32. To multiply $5+3$ by 4, we multiply 5 by 4, and then 3 by 4, and add the second result to the first.

Thus, $$(5+3)4=5\times4+3\times4.$$

We then assume that to multiply $a+b$ by c, we multiply a by c, and b by c, and add the second result to the first.

Thus, $$(a+b)c=ac+bc.$$

33. Addition of Similar Terms (§ 27).

1. Required the sum of $5a$ and $3a$.

We have, $5a+3a=(5+3)a$ (§ 32)

$=8a.$

That is, we do not add the a's but the coefficients of the a's.

2. Required the sum of $-5a$ and $-3a$.

We have, $(-5a)+(-3a)=(-5)\times a+(-3)\times a$ (§ 26)

$=[(-5)+(-3)]\times a$ (§ 32)

$=(-8)\times a$ (§ 15)

$=-8a.$ (§ 26)

3. Required the sum of $5a$ and $-3a$.

We have, $5a+(-3)a=[5+(-3)]\times a$ (§ 32)

$=2a.$ (§ 15)

4. Required the sum of $-5a$ and $3a$.

We have, $(-5)a+3a=[(-5)+3]\times a$ (§ 32)

$=(-2)\times a$ (§ 15) $=-2a.$

Therefore, to add two similar terms, find the sum of their numerical coefficients (§§ 15, 25, 26), and affix to the result the common letters.

5. Find the sum of $2a$, $-a$, $3a$, $-12a$, and $6a$.

Since the additions may be performed in any order, we may add the positive terms first, and then the negative terms, and finally combine these two results.

The sum of $2a$, $3a$, and $6a$ is $11a$.
The sum of $-a$ and $-12a$ is $-13a$.
Hence, the required sum is $11a+(-13a)$, or $-2a$.

6. Add $3(a-b)$, $-2(a-b)$, $6(a-b)$, and $-4(a-b)$.

The sum of $3(a-b)$ and $6(a-b)$ is $9(a-b)$.
The sum of $-2(a-b)$ and $-4(a-b)$ is $-6(a-b)$.
Then, the result is $[9+(-6)](a-b)$, or $3(a-b)$.

If the terms are not all similar, we may combine the similar terms, and unite the others with their respective signs (§ 30).

7. Required the sum of $12\,a$, $-5\,x$, $-3\,y^2$, $-5\,a$, $8\,x$, and $-3\,x$.

The sum of $12\,a$ and $-5\,a$ is $7\,a$.
The sum of $-5\,x$, $8\,x$, and $-3\,x$ is 0.
Then, the required sum is $7\,a-3\,y^2$.

EXERCISE 7

Add the following:

1. $8\,m$ and $4\,m$.
2. $12\,a$ and $-5\,a$.
3. $12\,a$ and $-16\,a$.
4. $-12\,a$ and $-5\,a$.
5. $-8\,y^2$ and $-20\,y^2$.
6. $-15\,cd$ and $13\,cd$.
7. $24\,a^2b$ and $-23\,a^2b$.
8. $31\,c^5d^2$ and $-31\,c^5d^2$.
9. $-6(c+d)$ and $-4(c+d)$.
10. $-5(x^2+y^2)$ and $-9(x^2+y^2)$.
11. $7\,x$, $4\,x$, and $-3\,x$.
12. $16\,a$, $-5\,a$, $-3\,a$, and a.
13. $2\,a$, $-5\,a$, and $-11\,a$.
14. $3\,xyz$, $6\,xyz$, and $-9\,xyz$.
15. $8(x+y)$, $-14(x+y)$, and $3(x+y)$.
16. $8\,n^2$, $-n^2$, $14\,n^2$, $-4\,n^2$, and $7\,n^2$.
17. $3\,a^4b^2$, $-5\,a^4b^2$, a^4b^2, $-9\,a^4b^2$, and $20\,a^4b^2$.
18. $3\,ax$, $-4\,bx$, $5\,ax$, and $-2\,bx$.
19. $3(a+b)$, $4(a-b)$, $-2(a+b)$, and $6(a-b)$.
20. $4\,h$, $3\,k$, $-5\,a$, $2\,k$, $-h$, and $2\,a$.

ADDITION OF POLYNOMIALS

34. A **Polynomial** is an algebraic expression consisting of more than one term; as $a+b$, or $2\,x^2-xy-3\,y^2$.

A polynomial is also called a *multinomial.*

A **Binomial** is a polynomial of *two* terms; as $a+b$.

A **Trinomial** is a polynomial of *three* terms; as $a+b-c$.

35. A polynomial is said to be *arranged* according to the *descending* powers of any letter, when the term containing the highest power of that letter is placed first, that having the next lower immediately after, and so on.

Thus, $$x^4+3x^3y-2x^2y^2+3xy^3-4y^4$$

is arranged according to the descending powers of x.

The term $-4y^4$, which does not involve x at all, is regarded as containing the lowest power of x in the above expression.

A polynomial is said to be arranged according to the *ascending* powers of any letter, when the term containing the lowest power of that letter is placed first, that having the next higher immediately after, and so on.

Thus, $$x^4+3x^3y-2x^2y^2+3xy^3-4y^4$$

is arranged according to the ascending powers of y.

36. Addition of Polynomials.

Let it be required to add $b+c$ to a.

Since $b+c$ is the sum of b and c (§ 29), we may add $b+c$ to a by adding b and c separately to a.

Then, $$a+(b+c)=a+b+c.$$

(To indicate the addition of $b+c$, we write it in parenthesis.)

The above assumes that, to add the sum of a set of terms, we add the terms separately.

37. From § 36 we have the following rule:

To add a polynomial to a quantity, add its terms with their signs unchanged.

1. Add $6a-7x^2$, $3x^2-2a+3y^3$, and $2x^2-a-mn$.

We set the expressions down one underneath the other, similar terms being in the same vertical column.

We then find the sum of the terms in each column, and write the results with their respective signs; thus,

$$\begin{array}{rrrr}
6a & -7x^2 & & \\
-2a & +3x^2 & +3y^3 & \\
-\ \ a & +2x^2 & & -mn \\
\hline
3a & -2x^2 & +3y^3 & -mn
\end{array}$$

2. Add $4x-3x^2-11+5x^3$, $12x^2-7-8x^3-15x$, and $14+6x^3+10x-9x^2$.

It is convenient to arrange each expression in *descending* powers of x (§ 35); thus,

$$\begin{array}{rrrr} 5x^3 & -\ 3x^2 & +\ 4x & -11 \\ -8x^3 & +12x^2 & -15x & -\ 7 \\ 6x^3 & -\ 9x^2 & +10x & +14 \\ \hline 3x^3 & & -\ x & -\ 4 \end{array}$$

3. Add $9(a+b)-8(b+c)$, $-3(b+c)-7(c+a)$, and
$4(c+a)-5(a+b)$.

$$\begin{array}{rrr} 9(a+b) & -\ 8(b+c) & \\ & -\ 3(b+c) & -7(c+a) \\ -5(a+b) & & +4(c+a) \\ \hline 4(a+b) & -11(b+c) & -3(c+a) \end{array}$$

4. Add $\frac{3}{4}a+\frac{2}{5}b-\frac{1}{3}c$ and $\frac{1}{6}a-\frac{4}{3}b+\frac{5}{7}c$.

$$\begin{array}{rrr} \frac{3}{4}a & +\ \frac{2}{5}b & -\ \frac{1}{3}c \\ \frac{1}{6}a & -\ \frac{4}{3}b & +\ \frac{5}{7}c \\ \hline \frac{11}{12}a & -\frac{14}{15}b & +\frac{8}{21}c \end{array}$$

EXERCISE 8

Add the following: (Results may be checked as in Chap. XVII.)

1.	2.	3.
$2a-5b$	$-\ 4x^2+3y^2$	$-8xy+2st$
$-7a+6b$	x^2-4y^2	$+2xy-7st$
$9a-\ b$	$-11x^2+8y^2$	$-3xy+5st$

4a.

4. $7d-4r-6n$ and $3d+9r+2n$.

5. $5a^2-4ab+b^2$, $4a^2+4ab+5b^2$, and $-9a^2+6b^2$.

6. $2m-3x+f$, $m+x-f$, and $m+f$.

7. $3bu+2pk-qt$, $5bu-7pk+2qt$.

8. $b-2+3b^2-8b^3$, $b+6-b^2+7b^3$, and $b+2b^2-4b^3$.

9. $3(a+b)-7(b+c)$, $5(a+b)+5(b+c)$, and
$-2(a+b)-3(b+c)$.

10. $\frac{1}{2}a-\frac{1}{3}b+\frac{2}{5}c$ and $-\frac{3}{4}a+\frac{4}{9}b-\frac{1}{10}c$.

11. $4t+3u-5c$, $-2t-a+3c$, $2a-9c+2u$, and
$5t+3a-4u$.

12. $\frac{8}{15}x+\frac{2}{9}y+\frac{13}{10}z$ and $\frac{8}{10}x-\frac{7}{9}y-\frac{1}{2}z$.

13. Add these equations (Ax. 5, § 4), then find the value of x:

$$\begin{cases} x+y=5, \\ x-y=7. \end{cases}$$

After x is known can you find y?

14. Find y by adding these equations:

$$\begin{cases} 5x+2y=16, \\ -5x+3y=-1. \end{cases}$$

What value has x?

15. Find x and y in these equations:

$$\begin{cases} 2x+3y=11, \\ x-3y=1. \end{cases}$$

Add the following:

16. $14(x+y)-17(y+z)$, $4(y+z)-9(z+x)$, and $-3(x+y)-7(z+x)$.

17. $6c+2a-3b$, $4d-7c+12a$, $8b-5d+c$, and $-10a-11b+9d$.

18. $-7(a-b)^2+8(a-b)+2$, $4(a-b)^2-5(a-b)$, and $3(a-b)^2-9$.

EXERCISE 9

	1st No.		2nd No.		Sum.		1st No.		2nd No.		Sum.
1.	$+8$	$+$	?	$=$	5	6.	x	$+$	?	$=$	$x+y$
2.	-10	$+$	?	$=$	-7	7.	a	$+$	?	$=$	$a-b$
3.	$+10$	$+$	?	$=$	-7	8.	a	$+$	?	$=$	b
4.	$+6$	$+$	?	$=$	11	9.	$-b$	$+$	?	$=$	a
5.	-3	$+$	?	$=$	-9	10.	c	$+$	?	$=$	b

38. In the above examples we have given the sum of two numbers and one of the numbers to find the other number.

SUBTRACTION OF MONOMIALS

39. Subtraction is the process of finding one of two numbers when their sum and one of them is given.

The **Minuend** is the sum of the numbers.

The **Subtrahend** is the given number.

The **Difference** is the required number.

40. Therefore, to subtract one number from another is to find a number, which added to the subtrahend will produce the minuend.

For example, to subtract 3 from 10, we find the number, which, added to 3, will produce 10. By remembering the result in addition such number is seen to be 7. Thus 7 is our *difference*.

To subtract -4 from 9, find the number which, added to -4, will produce 9. By inspection this number is evidently 13.

Subtract -6 from -8.

$$-6 \text{ plus } -2 \text{ gives } -8,$$

hence our *difference* is -2.

Subtract $+3$ from -9.

$$3+(-12)=-9,$$

hence -12 is our *difference*.

EXERCISE 10

Subtract the following:

1. 7 from 2.
2. 3 from -8.
3. -11 from -10.
4. -3 from 8.
5. -6 from -11.
6. 36 from 12.
7. 6 from 13.
8. -9 from 3.

9.	10.	11.	12.
$9x$	$4a$	$-4a$	$13t$
$3x$	$-5a$	$-7a$	$-\ t$

41. Notice that in each of the above examples the result is the same as if we had changed the sign of the subtrahend and proceeded as if adding the subtrahend to the minuend.

42. Similarly, from § 41 this rule follows:

To subtract one number from another, change the sign of the subtrahend and proceed as in addition. (The sign of the subtrahend must be changed *mentally*.)

EXERCISE 11

Subtract the following:

(The accuracy of all results may be checked by adding the difference to the subtrahend.)

1. $5ax$ from ax.
2. $3abc$ from $-9abc$.
3. $14a^2b^2$ from $11a^2b^2$.
4. $15(a-b)$ from $19(a-b)$.

5. $\frac{1}{3}\,my$ from $\frac{1}{2}\,my$.
6. $-11\,c^2s$ from $-6\,c^2s$.
7. $-21\,cy$ from $13\,cy$.
8. From $8\,a$ take $3\,b$.
9. From $7\,x$ take $-2\,y$.
10. From $-3\,a$ take $4\,b^2$.

SUBTRACTION OF POLYNOMIALS

43. Since a polynomial may be regarded as the sum of its separate terms (§ 30), we have the following rule:

To subtract a polynomial, change the sign of each of its terms, and add the result to the minuend.

1. Subtract $7\,ab^2-9\,a^2b+8\,b^3$ from $5\,a^3-2\,a^2b+4\,ab^2$.

It is convenient to place the subtrahend under the minuend, so that similar terms shall be in the same vertical column.

We then *mentally* change the sign of each term of the subtrahend, and add the result to the minuend; thus,

$$\begin{array}{r} 5\,a^3-2\,a^2b+4\,ab^2 \\ -9\,a^2b+7\,ab^2+8\,b^3 \\ \hline 5\,a^3+7\,a^2b-3\,ab^2-8\,b^3 \end{array}$$

2. Subtract the sum of $9\,x^2-8\,x+x^3$ and $5-x^2+x$ from $6\,x^3-7\,x-4$.

We change the sign of each expression which is to be subtracted, and add the results.

$$\begin{array}{rrrr} 6\,x^3 & & -7\,x & -4 \\ -\,x^3 & -9\,x^2 & +8\,x & \\ & +\,x^2 & -\,x & -5 \\ \hline 5\,x^3 & -8\,x^2 & & -9 \end{array}$$

EXERCISE 12

Subtract the following:

1.
$$\begin{array}{r} x^2+13\,x-11 \\ -3\,x^2+6\,x-5 \\ \hline \end{array}$$

2.
$$\begin{array}{r} -2\,m^2-4\,mn+9\,n^2 \\ 8\,m^2-7\,mn+14\,n^2 \\ \hline \end{array}$$

3.
$$\begin{array}{r} ab+bc+ca \\ ab-bc+ca \\ \hline \end{array}$$

4. From $9\,a+4\,h-5\,k$ take $9\,a-4\,h+5\,k$.
5. From $6\,x^3-5\,x^2+4\,x-3$ take $x^3-3\,x^2-2\,x+1$.
6. From $11\,a-9\,b+2\,z$ subtract $-3\,z+2\,a-14\,b$.

7. Take $-8(h+k)+3(h-k)$ from $(h+k)-4(h-k)$.

8. Subtract $74\,z^2-47\,zk+30\,k^2$ from $24\,k^2-30\,zk+10\,z^2$.

9. From $7\,v-8\,s+75\,t^2$ take $-16\,v+19\,s$.

10. What must be added to $4\,g+18\,z^2-x$ to give 0?

11. By how much does $8\,x^3-7\,x^2+5\,x-1$ exceed
$x^3+14\,x^2-3\,x+7$?

12. From $x^3-11\,x+4$ subtract $8\,x^2-3\,x-1$.

13. From $a^2+2\,ab+b^2$ take $a^2-2\,ab+b^2$.

14. Find the sum of $a^2+2\,ab+b^2$ and $a^2-2\,ab+b^2$.

15. From the sum of $x^3+4\,x^2+4\,x$ and $2\,x^2+8\,x+8$ take $6\,x^2+12\,x$.

16. From the sum of $a^3-2\,a^2b+ab^2$ and $-a^2b+2\,ab^2-b^3$ take the sum of $a^3+2\,a^2b+ab^2$ and $a^2b+2\,ab^2+b^3$.

17. From $\frac{4}{5}\,s-\frac{5}{6}\,a+\frac{2}{3}\,b$ take $\frac{1}{5}\,s+\frac{1}{2}\,a-\frac{1}{6}\,b$.

18. From $\frac{3}{2}\,gt^2+v-\frac{1}{3}\,t$ take $gt^2-\frac{4}{5}\,v+6\,t$.

19. Take $a^4-6\,a^3-15\,a^2-8\,a+4$
from $7\,a^4+3\,a^3-5\,a^2-11\,a-9$.

20. From $\frac{1}{2}\,m-\frac{1}{3}\,n+\frac{2}{7}\,p$ take $\frac{3}{5}\,m-\frac{3}{4}\,n+\frac{7}{3}\,p$.

21. From $n^4-10\,n^3x-n^2x^2+8\,nx^3+3\,x^4$
take $5\,n^4+4\,n^3x-9\,n^2x^2+2\,nx^3-12\,x^4$.

22. Take $18\,x^4-8\,x+6\,x^5+12-8\,x^3$
from $-10\,x^3+2-15\,x^2+11\,x^5-4\,x$.

23. Take $a^5-10\,a^3b^2+13\,a^2b^3-7\,ab^4-5\,b^5$
from $9\,a^5+3\,a^4b+6\,a^3b^2-a^2b^3-16\,b^5$.

24. From the sum of $2\,x^2-5\,xy-y^2$ and $7\,x^2-3\,xy+9\,y^2$
subtract $4\,x^2-6\,xy+8\,y^2$.

25. From 0 subtract the sum of $4\,a^2$ and $3\,a-5\,a^2-1$.

Add the following pairs of equations to find x, subtract them (Ax. 6, § 4) to find y. Verify results by substituting the values of x and y in the given equations:

26. $\begin{cases} x+y=5, \\ x-y=1. \end{cases}$ 27. $\begin{cases} 2\,x+5\,y=16, \\ 2\,x-5\,y=-4. \end{cases}$ 28. $\begin{cases} 5\,y+x=9, \\ 5\,y-x=1. \end{cases}$

PARENTHESES

44. Removal of Parentheses.

By § 30, $a+(b-c)=a+b-c.$ Hence,

Parentheses preceded by a + sign may be removed without changing the signs of the terms enclosed.

Again, by § 43, $a-(b-c)=a-b+c.$ Hence,

Parentheses preceded by a − sign may be removed if the sign of each term enclosed be changed.

The above rules apply equally to the removal of the *brackets*, *braces*, or *vinculum* (§ 7).

It should be noticed in the case of the latter that the sign apparently prefixed to the first term underneath is in reality prefixed to the vinculum; thus, $+\overline{a-b}$ means the same as $+(a-b)$, and $-\overline{a-b}$ the same as $-(a-b)$.

45. 1. Remove the parentheses from

$$2a-3b-(5a-4b)+(4a-b).$$

By the rules of § 44, the expression becomes

$$2a-3b-5a+4b+4a-b=a.$$

Parentheses sometimes enclose others; in this case they may be removed in succession by the rules of § 44.

Beginners should remove one at a time, commencing with the *innermost* pair; after a little practice, they should be able to remove several signs of aggregation at one operation, in which case they should commence with the outermost pair.

2. Simplify $4x-\{3x+(-2x-\overline{x-a})\}$.

We remove the vinculum first, then the parentheses, and finally the braces.

Thus,
$$\begin{aligned} &4x-\{3x+(-2x-\overline{x-a})\}\\ &=4x-\{3x+(-2x-x+a)\}\\ &=4x-\{3x-2x-x+a\}\\ &=4x-3x+2x+x-a=4x-a. \end{aligned}$$

EXERCISE 13

1. What is the sign of $2x$ in $3x^2-4c-(2x+1)$?

2. What is the sign of a in $4a^3-\overline{a-4c^2}+9x^2$? What is the coefficient of a after the vinculum is removed?

Simplify by removing the signs of aggregation and then uniting similar terms:

3. $11\,a-(-6\,m+5\,c)-(3\,a+4\,c)$.
4. $4\,x-3\,y-[7\,y-d]+\{-4\,x-3\,y\}$.
5. $x^2+[-3\,x^2-(2\,y^2-2\,x^2)+2\,y^2]$.
6. $7\,t+u-\{6\,t-\overline{u+7}-3\}$.
7. $(a^2+2\,ab+b^2)-(a^2-2\,ab+b^2)$.

Compare Ex. 13, Exercise 12.

8. $x^3-(-3\,x^2y-3\,xy^2)+y^3-(x^3-[3\,x^2y-3\,xy^2+y^3])$.

Compare Ex. 16, Exercise 12.

9. $7\,x-\{-8\,y-\overline{10\,x-11\,y}\}$.
10. $a^3-(-6\,a^2-\overline{12\,a+8})-(a^3+12\,a)$.

Compare Ex. 15, Exercise 12.

46. Insertion of Parentheses. — To write terms in parenthesis, we take the converse of the rules of § 44.

Any number of terms may be written in parenthesis preceded by a + sign, without changing their signs.

Any number of terms may be written in parenthesis preceded by a − sign, if the sign of each term be changed.

Ex. Write the last three terms of $a-b+c-d+e$ in parenthesis preceded by a − sign.

Result, $a-b-(-c+d-e)$.

EXERCISE 14

In each of the following expressions, write the last three terms in parenthesis preceded by a − sign:

1. $a+b+c-d$.
2. $m^2+3\,m-2+h$.
3. $x^3-3\,x^2+3\,x-1$.
4. $4\,a^4-3\,a^3-2\,a^2-a$.
5. $8\,x^2-y^2-y+z$.
6. $a^2-b^2+c^2-d^2$.
7. $x^2-2\,xy-y^2-2\,yz-z^2$.
8. $2\,a^5-10\,a^4-8\,a^3+5\,a^2-6\,a+9$.
9. In each of the above results, write the last two terms in parenthesis in brackets preceded by a − sign.

47. Addition and Subtraction of Terms having Literal Coefficients. — To add two or more terms involving the same power of a certain letter, with literal, or numerical and literal, coefficients, it is convenient to put the coefficient of this letter in parenthesis.

1. Add ax and $2x$.

By § 32, $$ax+2x=(a+2)x.$$

2. Add $(2m+n)y$ and $(m-3n)y$.

$$(2m+n)y+(m-3n)y=[(2m+n)+(m-3n)]y$$
$$=(2m+n+m-3n)y=(3m-2n)y.$$

(The pupil should endeavor to put down the result in one operation.)

3. Subtract $(b-c)x^2$ from ax^2.

By §§ 32, 42, 44, $$ax^2-(b-c)x^2=[a-(b-c)]x^2$$
$$=(a-b+c)x^2.$$

EXERCISE 15

Add the following:

1. ax and bx.
2. $3ab^2$ and $-4b^2$.
3. $-a^2b$ and $5a^2y$.
4. $3a^2bc$ and $-8cd$.
5. ab, bc and $-m^2b$.
6. c^2x^3 and $(a-3d)x^3$.
7. $(7a+4b)x^2$ and $(3m+n)x^2$.
8. $(4x-y)z^4$ and $(3x+c)z^4$.

Subtract the following:

9. $3cx$ from dx.
10. $-4my$ from $3cy$.
11. $-cxy$ from $-dxy$.
12. $(c+d)x$ from ax.
13. $(3c-4d)x^2$ from $(6c+9d)x^2$.

QUERIES

1. What is the difference between $4x$ and $3y$?

2. Express the sum of three times a certain number and four times the same number.

3. Do you make any distinction between a *factor* and a *coefficient?*

4. Regarding $3mxy$ and $5cdx$ as the coefficients of the expressions $3amxy$ and $5acdx$, find the sum. Regard ax as the common part and find the sum.

5. Does the sum of a and b, $(a+b)$, have the same significance to you as the sum of 5 and 2, $(5+2)$?

6. Express algebraically: the sum of 6 times the sum of a and b, and 9 times the sum of the same two numbers.

7. What must you add to a polynomial to produce 0? Give an example.

8. Did you work problem 7 by addition or subtraction? Is there any difference between the two processes as here used?

9. Have you noticed which forms of the signs of aggregation are used most? Does the vinculum appear often?

10. Subtract $3\,m(a+b)$ from $6\,m(a-b)$.

11 If your difference is $x^2-4\,x+7$ and your subtrahend is $3\,x^2+4\,x-9$, what is your minuend?

IV. MULTIPLICATION OF ALGEBRAIC EXPRESSIONS

48. The Rule of Signs.

If a and b are any two positive numbers, we have by § 20,

$$(+a)\times(+b)=+ab, \qquad (+a)\times(-b)=-ab,$$
$$(-a)\times(+b)=-ab, \qquad (-a)\times(-b)=+ab.$$

From these results we may state what is called the **Rule of Signs** in multiplication, as follows:

The product of two terms of like sign is positive; the product of two terms of unlike sign is negative.

49. We have by § 48,

$$\begin{aligned}(-a)\times(-b)\times(-c)&=(ab)\times(-c)\\&=-abc;\end{aligned} \tag{1}$$

$$\begin{aligned}(-a)\times(-b)\times(-c)\times(-d)&=(-abc)\times(-d), \text{ by (1)},\\&=abcd; \text{ etc.}\end{aligned}$$

That is, the product of three negative terms is negative; the product of four negative terms is positive; and so on.

In general, **the product of any number of terms is positive or negative according as the number of negative terms is even or odd.**

50. The Law of Exponents.

Let it be required to multiply a^3 by a^2.

By § 6, $a^3 = a \times a \times a,$

and $a^2 = a \times a.$

Whence, $a^3 \times a^2 = a \times a \times a \times a \times a = a^5.$

The general case. — Let it be required to multiply a^m by a^n, where m and n are any positive integers.

We have $a^m = a \times a \times \cdots$ to m factors,

and $a^n = a \times a \times \cdots$ to n factors.

Then, $a^m \times a^n = a \times a \times \cdots$ to $m+n$ factors $= a^{m+n}.$

(The *Sign of Continuation*, $\cdots$, is read "*and so on.*")

Hence, the exponent of a letter in the product is equal to its exponent in the multiplicand plus its exponent in the multiplier.

This is called the *Law of Exponents for Multiplication.*

A similar result holds for the product of three or more powers of the same letter.

Thus, $a^3 \times a^4 \times a^5 = a^{3+4+5} = a^{12}.$

MULTIPLICATION OF MONOMIALS

51. 1. Let it be required to multiply $7a$ by $-2b$.

By § 26, $-2b = (-2) \times b.$

Then, $7a \times (-2b) = 7a \times (-2) \times b$

$= 7 \times (-2) \times a \times b = -14ab.$ (§ 48)

In the above solution, we assume that *the factors of a product can be written in any order.*

2. Required the product of $-2a^2b^3$, $6ab^5$, and $-7a^4c$.

$$(-2a^2b^3) \times 6ab^5 \times (-7a^4c)$$
$$= (-2)a^2b^3 \times 6ab^5 \times (-7)a^4c$$
$$= (-2) \times 6 \times (-7) \times a^2 \times a \times a^4 \times b^3 \times b^5 \times c$$
$$= 84a^7b^8c, \text{ by §§ 49 and 50.}$$

We then have the following rule for the product of any number of monomials:

To the product of the numerical coefficients (§§ 25, 26, 49, 50) annex the letters; giving to each an exponent equal to the sum of its exponents in the factors.

3. Multiply $-5\,a^3b$ by $-8\,ab^3$.

$$(-5\,a^3b)\times(-8\,ab^3)=40\,a^{3+1}b^{1+3}=40\,a^4b^4.$$

4. Find the product of $4\,n^2$, $-3\,n^5$, and $2\,n^4$.

$$4\,n^2\times(-3\,n^5)\times 2\,n^4=-24\,n^{2+5+4}=-24\,n^{11}.$$

5. Multiply $-x^m$ by $7\,x^6$.

$$(-x^m)\times 7\,x^6=-7\,x^{m+6}.$$

6. Multiply $6(m+n)^4$ by $7(m+n)^3$.

$$6\,(m+n)^4\times 7\,(m+n)^3=42\,(m+n)^7.$$

EXERCISE 16

Multiply the following:

1. $4\,a^2$ by $7\,a^5$.
2. $6\,m^2d$ by $9\,md^2$.
3. $11\,ax$ by $-8\,ab$.
4. $-7\,cx$ by $-10\,rs$.
5. $-12\,x^3y$ by $9\,xy^3$.
6. $4\,x^ay^2z$ by $-11\,x^2y^mz^3$.
7. $3\,(x+y)^2$ by $16\,(x+y)^3$.
8. $-4\,(a-b)$ by $6\,(a-b)^2$.

Find the product of:

9. $(x-y)^2$, $4(x-y)^3$, and $-7\,(x-y)$.
10. $3\,m^2nz^3$, $-5\,mn^3z$, and $-6\,mnz^4$.
11. a^4c^2, $-2\,a^3c^4$, $-5\,ac^5$, and $6\,a^5bc$.
12. a^xb^y, $4\,a^2b$, and $-11\,a^{2x}b^{3y}$.

MULTIPLICATION OF POLYNOMIALS BY MONOMIALS

52. In § 32, we assumed that the product of $a+b$ by c was $ac+bc$. We then have the following rule for the product of a polynomial by a monomial:

Multiply each term of the multiplicand by the multiplier, and add the partial products.

Ex. Multiply $2\,x^2-5\,x+7$ by $-8\,x^3$.

$$\begin{aligned}&(2\,x^2-5\,x+7)\times(-8\,x^3)\\&=(2\,x^2)\times(-8\,x^3)+(-5\,x)\times(-8\,x^3)+(7)\times(-8\,x^3)\\&=-16\,x^5+40\,x^4-56\,x^3.\end{aligned}$$

The written work should stand as follows:

$$\begin{array}{r}2\,x^2-\ 5\,x+\ 7\\ \underline{-\ 8\,x^3\qquad\qquad}\\ -16\,x^5+40\,x^4-56\,x^3\end{array}$$

EXERCISE 17

Multiply the following:

1. $12a-3$ by $5a$.
2. $3a^2b+7ab^2$ by $-9a^2b^2$.
3. $x^4-8x^3y^2+8xy^3$ by $-x^2y^2$.
4. $4m^4-3n^2-4$ by $7m^3$.
5. $5z^2$ by $8z^4-3z^2+13$.
6. $-9cd^3$ by $5c^3-10c^2d+2d^3$.
7. $8a^5-4a^6+9a^7$ by $-8a^4$.
8. $8x^n-3x^{2n}$ by $15x^{3n}$.
9. $-15a^2b+7b^2-4a^4$ by $-8a^5b^2$.
10. $4x^2y^2$ by $x^3+6x^2y-6xy^2+8y^3$.
11. $12a^2b-6ab^2-8b^3+a^3$ by $-8ab$.
12. $6a^3-8a^2c+7ac^2-11c^3$ by $14a^4c^3$.
13. $a^3-9a^2b+27ab^2-27b^3$ by $-3b$.
14. $\frac{1}{4}a^2-\frac{1}{3}ab+\frac{1}{9}b^2$ by $-\frac{1}{3}b$.
15. $\frac{8}{15}c^2-\frac{4}{5}b^2c-\frac{3}{10}c^3d$ by $5cd$.

MULTIPLICATION OF POLYNOMIALS BY POLYNOMIALS

53. Let it be required to multiply $a+b$ by $c+d$.

As in § 32, we multiply $a+b$ by c, and then $a+b$ by d, and add the second result to the first; that is,

$$\begin{aligned}(a+b)(c+d)&=(a+b)c+(a+b)d\\&=ac+bc+ad+bd.\end{aligned}$$

RULE: — **Multiply each term of the multiplicand by each term of the multiplier, and add the partial products.**

54. 1. Multiply $3a-4b$ by $2a-5b$.

In accordance with the rule, we multiply $3a-4b$ by $2a$, and then by $-5b$, and add the partial products.

A convenient arrangement of the work is shown below, similar terms of the partial products being in the same vertical column.

$$\begin{array}{l} 3a-4b \\ \underline{2a-5b} \\ 6a^2-\ \ 8ab \\ \underline{\qquad -15ab+20b^2} \\ 6a^2-23ab+20b^2 \end{array}$$

The work may be checked by solving the example with the multiplicand and multiplier interchanged.

2. Multiply $4ax^2+a^3-8x^3-2a^2x$ by $2x+a$.

It is convenient to arrange the multiplicand and multiplier in the same order of powers of some common letter (§ 35), and write the partial products in the same order.

Arranging the expressions according to the descending powers of a, we have

$$\begin{array}{l} a^3-2a^2x+4ax^2-8x^3 \\ \underline{a+2x} \\ a^4-2a^3x+4a^2x^2-8ax^3 \\ \underline{\qquad 2a^3x-4a^2x^2+8ax^3-16x^4} \\ a^4 \qquad\qquad\qquad\qquad\qquad -16x^4 \end{array}$$

EXERCISE 18

Multiply the following:

1. $3x-4$ by $8x+5$.
2. $4a+b$ by $4a+b$.
3. $7t-4u$ by $11t-3u$.
4. $6ab+b^2$ by $3ab-5b^2$.
5. a^2-a+1 by $a+1$.
6. k^2-k-12 by $k-7$.
7. $2x+3$ by $2x-3$.
8. $3x+7$ by $3x-7$.
9. $\frac{1}{3}c+\frac{1}{4}d$ by $\frac{1}{3}c-\frac{1}{4}d$.
10. $\frac{1}{3}c+\frac{1}{4}d$ by $\frac{1}{3}c+\frac{1}{4}d$.
11. $\frac{1}{3}c-\frac{1}{4}d$ by $\frac{1}{3}c-\frac{1}{4}d$.

(Note carefully in what way examples 9 to 11, and 12 to 14 differ.)

12. $x+3$ by $x+5$.
13. $x-3$ by $x+5$.
14. $x+3$ by $x-5$.
15. a^2+a+1 by a^2-a+1.
16. $6(m+n)^2-5(m+n)+1$ by $7(m+n)-2$.
17. $3(a-b)^4-2(a-b)$ by $4(a-b)^2+(a-b)$.
18. $7t^2+8t-1$ by $7t^2-8t+1$.
19. $6ab+a^2+9$ by $3a^2-4+2ab$.

20. $2h^2-10h+5$ by $h^2+5h-10$.

21. $t^3-t^2u+tu^2-u^3$ by $t+u$.

22. $h^2-3hk+9k^2$ by $h+3k$.

23. x^2+xy+y^2 by $x-y$.

Note the similarity in examples 21–23.

24. $3a^x+7b^y$ by $3a^x-7b^y$.

25. $5(x+y)+7(x-y)$ by $3(x+y)-2(x-y)$.

26. e^x+e^y by e^x-e^y.

27. $a^3+2a^2b+2ab^2+b^3$ by $a^2-2ab+b^2$.

28. $4a^{m+5}b^2-3a^4b^c$ by $a^{m+2}b-2ab^{c-1}$.

29. $5x^4-6x^3-4x^2+2x-3$ by $3x-2$.

30. $a^3-3a^2x+3ax^2-x^3$ by $a^3+3a^2x+3ax^2+x^3$.

55. If the product has more than one term involving the same power of a certain letter, with literal, or numerical and literal, coefficients, we may put the coefficient of this letter in parentheses, as in § 47.

Ex. Multiply $x^2-ax-bx+ab$ by $x-a$.

$$\begin{array}{llll}
x^2-ax & -bx & +ab & \\
x-a & & & \\
\hline
x^3-ax^2 & -bx^2 & +abx & \\
\quad -ax^2 & & +a^2x+abx & -a^2b \\
\hline
x^3-(2a+b)x^2 & +(a^2+2ab)x & -a^2b &
\end{array}$$

As in § 47, $-2ax^2-bx^2$ is equivalent to $-(2a+b)x^2$, and $a^2x+2abx$ to $(a^2+2ab)x$.

EXERCISE 19

Multiply the following:

1. $x^2+ax+bx+ab$ by $x+c$.
2. $x^2-mx+nx-mn$ by $x-p$.
3. $x^2-bx-cx+bc$ by $x-a$.
4. $x^2+ax-bx-3ab$ by $x+b$.
5. $x^2+ax+2bx+2ab$ by $x-c$.
6. $x^2+px-5qx-5pq$ by $x-r$.
7. $x^2-3ax-bx+3ab$ by $x+2a$.
8. $x^2-4mx+nx-4mn$ by $x+3n$.

9. $x^2+3\,ax-2\,bx+6\,ab$ by $x-4\,c$.
10. $(a-b)x-3\,ab$ by $2\,x-(a-b)$.
11. $x^{2n}-5\,ax^n+4\,bx^n-2\,ab$ by x^n+c.
12. $(2\,a-1)x^2+(a+2)x-(a+3)$ by $(a-2)x-a$.

56. *Ex.* Simplify $(a-2\,x)^2-2(3\,a+x)(a-x)$.

To simplify the expression, we first multiply $a-2\,x$ by itself (§ 6); we then find the product of 2, $3\,a+x$, and $a-x$, and subtract the second result from the first.

$$\begin{array}{l} a-2\,x \\ \underline{a-2\,x} \\ a^2-2\,ax \\ \underline{\quad -2\,ax+4\,x^2} \\ a^2-4\,ax+4\,x^2 \end{array} \qquad\qquad \begin{array}{r} 3\,a+x \\ \underline{a-x} \\ 3\,a^2+\ ax \\ \underline{-3\,ax-\ x^2} \\ 3\,a^2-2\,ax-\ x^2 \\ \underline{2} \\ 6\,a^2-4\,ax-2\,x^2 \end{array}$$

Subtracting the second result from the first, we have

$$a^2-4\,ax+4\,x^2-6\,a^2+4\,ax+2\,x^2=-5\,a^2+6\,x^2.$$

EXERCISE 20

Simplify the following:

1. $(3\,a+5)(2\,a-8)+(4\,a-7)(a+6)$.
2. $(3\,x+2)(4\,x+3)-(3\,x-2)(4\,x-3)$.
3. $(a-2\,x)(b+3\,y)+(a+2\,x)(b-3\,y)$.
4. $(3\,m+1)^2(3\,m-1)^2$.
5. $(x-y)(x^2-y^2)-(x+y)(x^2+y^2)$.
6. $(2\,a+3\,b)^2-4(a-b)(a+5\,b)$.
7. $[3\,x-(5\,y+2\,z)]\,[3\,x-(5\,y-2\,z)]$.
8. $[m+2\,n-(2\,m-n)]\,[2\,m+n-(m-2\,n)]$.
9. $(a+b+c)^2-(a-b-c)^2$.
10. $(a+2)(a+3)(a-4)+(a-2)(a-3)(a+4)$.
11. $(\frac{5}{2}\,x-\frac{4}{3}\,y+\frac{1}{4}\,z)^2$.
12. $[2\,x^2+(3\,x-1)(4\,x+5)]\,[5\,x^2-(4\,x+3)(x-2)]$.
13. $(a+2\,b-c-3\,d)^2$.

14. $(a-2)(a+3)-(a-3)(a+4)-(a-4)(a+5)$.

15. $(x+2)(2x-1)(3x-4)-(x-2)(2x+1)(3x+4)$.

DEFINITIONS

57. A monomial is said to be *rational and integral* when it is either a number expressed in Arabic numerals, or a single letter with unity for its exponent, or the product of two or more such numbers or letters.

Thus, $3a^2b^3$, being equivalent to $3 \cdot a \cdot a \cdot b \cdot b \cdot b$, is rational and integral.

A polynomial is said to be rational and integral when each term is rational and integral; as $2x^2 - \frac{3}{4}ab + c^3$.

58. If a term has a literal portion which consists of a single letter with unity for its exponent, the term is said to be of the *first degree*. Thus, $2a$ is of the first degree.

The *degree* of any rational and integral monomial (§ 57) is the number of terms of the first degree which are multiplied together to form its literal portion.

Thus, $5ab$ is of the *second* degree; $3a^2b^3$, being equivalent to $3 \cdot a \cdot a \cdot b \cdot b \cdot b$, is of the *fifth* degree; etc.

The degree of a rational and integral monomial equals the sum of the exponents of the letters involved in it.

Thus, ab^4c^3 is of the *eighth* degree.

The degree of a rational and integral polynomial is the degree of its term of highest degree.

Thus, $2a^2b - 3c + d^2$ is of the *third* degree.

Frequently the degree of a term or polynomial with respect to some particular letter is required. Thus $3a^2x^3 - 4bxy^5 + 2c^4$ is of the *third* degree in x.

59. Homogeneity. — *Homogeneous terms* are terms of the same degree. Thus, a^4, $3b^3c$, and $-5x^2y^2$ are homogeneous terms.

A polynomial is said to be homogeneous when its terms are homogeneous; as $a^3 + 3b^2c - 4xyz$.

V. DIVISION OF ALGEBRAIC EXPRESSIONS

60. **Division**, in Algebra, is the process of finding one of two numbers, when their product and the other number are given.

The **Dividend** is the product of the numbers.

The **Divisor** is the given number.

The **Quotient** is the required number.

61. The Rule of Signs. — Since the dividend is the product of the divisor and quotient, the equations of § 48 may be written as follows:

$$\frac{+ab}{+a}=+b,\quad \frac{-ab}{-a}=+b,\quad \frac{-ab}{+a}=-b,\ \text{and}\ \frac{+ab}{-a}=-b.$$

We may state the Rule of Signs in division as follows:

The quotient of two terms of like sign is positive; the quotient of two terms of unlike sign is negative.

62. Let
$$\frac{a}{b}=x. \qquad (1)$$

Then, since the dividend is the product of the divisor and quotient, we have
$$a=bx.$$

Multiply each of these equals by c (Ax. 7, § 4),
$$ac=bcx.$$

Regarding ac as the dividend, bc as the divisor, and x as the quotient, this may be written
$$\frac{ac}{bc}=x. \qquad (2)$$

From (1) and (2), $\frac{ac}{bc}=\frac{a}{b}$. (Ax. 4, § 4)

That is, a factor common to the dividend and divisor can be removed, or cancelled.

63. The Law of Exponents for Division. — Let it be required to divide a^5 by a^2.

By § 6,
$$\frac{a^5}{a^2}=\frac{a\times a\times a\times a\times a}{a\times a}.$$

Cancelling the common factor $a\times a$ (§ 62), we have

$$\frac{a^5}{a^2}=a\times a\times a=a^3.$$

Consider the general case:

Let it be required to divide a^m by a^n, where m and n are any positive integers such that m is greater than n.

We have, $$\frac{a^m}{a^n}=\frac{a\times a\times a\times\cdots\text{ to } m \text{ factors}}{a\times a\times a\times\cdots\text{ to } n \text{ factors}}.$$

Cancelling the common factor $a\times a\times a\times\cdots$ to n factors,

$$\frac{a^m}{a^n}=a\times a\times a\times\cdots\text{ to } m-n \text{ factors}=a^{m-n}.$$

Hence, the exponent of a letter in the quotient is equal to its exponent in the dividend, minus its exponent in the divisor.

This is called the *Law of Exponents for Division.*

V

DIVISION OF MONOMIALS

64. 1. Let it be required to divide $-14a^2b$ by $7a^2$.

By § 51, $$\frac{-14a^2b}{7a^2}=\frac{(-2)\times 7\times a^2\times b}{7\times a^2}.$$

Cancelling the common factors 7 and a^2 (§ 62), we have

$$\frac{-14a^2b}{7a^2}=(-2)\times b=-2b.$$

Then to find the quotient of two monomials:

To the quotient of the numerical coefficients annex the letters, giving to each an exponent equal to its exponent in the dividend minus its exponent in the divisor, and omitting any letter having the same exponent in the dividend and divisor.

The work may be checked by multiplying the divisor by the quotient.

2. Divide $54a^5b^3c^2$ by $-9a^4b^3$.

$$\frac{54a^5b^3c^2}{-9a^4b^3}=-6a^{5-4}c^2=-6ac^2.$$

3. Divide $-2x^{2m}y^nz^r$ by $-x^my^nz^5$.

$$\frac{-2x^{2m}y^nz^r}{-x^my^nz^5}=2x^{2m-m}z^{r-5}=2x^mz^{r-5}.$$

4. Divide $35(a-b)^7$ by $7(a-b)^4$.

$$\frac{35(a-b)^7}{7(a-b)^4}=5(a-b)^3.$$

EXERCISE 21

Divide the following:

1. 30 by -5.
2. -42 by 6.
3. -48 by -4.
4. -64 by 8.
5. -135 by -9.
6. 176 by -11.
7. $-\frac{12}{35}$ by $\frac{8}{15}$.
8. $21a^{10}$ by $3a^7$.
9. $-63m^4n^6$ by $7m^2n^3$.
10. $5x^3y^2$ by $-x^3y$.
11. $12(c+d)^6$ by $3(c+d)^4$.
12. a^3b^2c by $-abc$.
13. $60(a-b)^8$ by $12(a-b)$.
14. $8(2m+3)$ by $4(2m+3)$.
15. $75x^5y^6$ by $-15x^5y^6$.
16. $81ab^3c$ by $27b^2c$.
17. $\frac{4}{5}x^2y$ by $\frac{1}{5}xy$.
18. $-\frac{5}{3}a^3b^3c^4$ by $-5ab^2$.
19. $6(a+b)^5$ by $3(a+b)^2$.

Find the numerical value when $a=2$, $b=-4$, $c=5$, and $d=-3$ of:

20. $\dfrac{7a+14b-12c}{13a-9b+17c}$.

21. $\dfrac{2a-b}{c-5d}-\dfrac{a+4b}{3c+d}$.

DIVISION OF POLYNOMIALS BY MONOMIALS

65. We have, from § 32,

$$(a+b)c=ac+bc.$$

Since the dividend is the product of the divisor and quotient (§ 60), we may regard $ac+bc$ as the dividend, c as the divisor, and $a+b$ as the quotient.

Whence, $$\frac{ac+bc}{c}=a+b.$$

Hence, to divide a polynomial by a monomial, we divide each term of the dividend by the divisor, and add the results.

Ex. Divide $9a^2b^2-6a^4c+12a^3bc^3$ by $-3a^2$.

$$\frac{9a^2b^2-6a^4c+12a^3bc^3}{-3a^2}=-3b^2+2a^2c-4abc^3.$$

EXERCISE 22

Divide the following:

1. $30a^4 - 25a^3 + 20a^2$ by $5a$.
2. $-33am^2 + 22a^4m$ by $11am$.
3. $18s^2t^3z - 24s^3t^4z^3 - 12st^2z^2$ by $-3st^2z$.
4. $72h^3k^2x^5 - 60hk^2x^3$ by $-12hx^3$.
5. $\frac{5}{6}a^3b^2 - \frac{2}{3}a^2b^3$ by $\frac{1}{3}ab$.
6. $9(a+b)^2 - 6(a+b)$ by $3(a+b)$.
7. $x^{m+2} + 2x^{2m+1} - 3x^{2m+2}$ by x^{m+1}.
8. $36a^{15} + 28a^{12} - 4a^9 - 20a^6$ by $4a^6$.
9. $45(a-b)^5 - 40(a-b)^4 + 25(a-b)^3$ by $5(a-b)^2$.
10. $8m^4n - 24m^2n^3 + 12m^5 - 31m^2n^2$ by $6m^2$.
11. $(x+y)^3 - 9(x+y)^2 + 27(x+y)$ by $(x+y)$.
12. $\frac{4}{9}m^4 - 2m^3 + \frac{9}{4}m^2$ by $\frac{2}{3}m^2$.
13. $15a^3 - 4a^5 + 8a^6 - 5a - 2a^4 + 3a^2$ by $-2a$.
14. $a^2(2m+3)^3 + 2ab(2m+3)^2 + b^2(2m+3)$ by $(2m+3)$.

DIVISION OF POLYNOMIALS BY POLYNOMIALS

66. Let it be required to divide $12 + 10x^3 - 11x - 21x^2$ by $2x^2 - 4 - 3x$.

Arranging each expression according to the descending powers of x (§ 35), we are to find an expression which, when multiplied by the divisor, $2x^2 - 3x - 4$, will produce the dividend, $10x^3 - 21x^2 - 11x + 12$.

It is evident that the term containing the highest power of x in the product is the product of the terms containing the highest powers of x in the multiplicand and multiplier.

Therefore, $10x^3$ is the product of $2x^2$ and the term containing the highest power of x in the quotient.

Whence, the term containing the highest power of x in the quotient is $10x^3$ divided by $2x^2$, or $5x$.

Multiplying the divisor by $5x$, we have the product $10x^3 - 15x^2 - 20x$;

which, when subtracted from the dividend, leaves the remainder $-6x^2+9x+12$.

This remainder must be the product of the divisor by the rest of the quotient; therefore, to obtain the next term of the quotient, we regard $-6x^2+9x+12$ as a new dividend.

Dividing the term containing the highest power of x, $-6x^2$, by the term containing the highest power of x in the divisor, $2x^2$, we obtain -3 as the second term of the quotient.

Multiplying the divisor by -3, we have the product $-6x^2+9x+12$; which, when subtracted from the second dividend, leaves no remainder.

Hence, $5x-3$ is the required quotient.

$$
\begin{array}{r|l}
10x^3-21x^2-11x+12 & 2x^2-3x-4, \text{ Divisor.} \\
\underline{10x^3-15x^2-20x} & 5x-3, \text{ Quotient.} \\
-\ 6x^2+\ 9x+12 & \\
\underline{-\ 6x^2+\ 9x+12} &
\end{array}
$$

The example might have been solved by arranging the dividend and divisor according to the *ascending* powers of x.

From the above example, we derive the following rule:

Arrange the dividend and divisor in the same order of powers of some common letter.

Divide the first term of the dividend by the first term of the divisor, and write the result as the first term of the quotient.

Multiply the whole divisor by the first term of the quotient, and subtract the product from the dividend.

If there be a remainder, regard it as a new dividend, and proceed as before; arranging the remainder in the same order of powers as the dividend and divisor.

1. Divide $9ab^2+a^3-9b^3-5a^2b$ by $3b^2+a^2-2ab$.

Arranging according to the descending powers of a,

$$
\begin{array}{l|l}
a^3-5a^2b+9ab^2-9b^3 & a^2-2ab+3b^2 \\
\underline{a^3-2a^2b+3ab^2} & a-3b \\
\quad -3a^2b+6ab^2 & \\
\quad \underline{-3a^2b+6ab^2-9b^3} &
\end{array}
$$

In the above example, the last term of the second dividend is omitted, as it is merely a repetition of the term directly above.

The work may be verified by multiplying the divisor by the quotient, which should of course give the dividend.

2. Divide $4+9x^4-28x^2$ by $-3x^2+2+4x$.

Arranging according to the ascending powers of x,

$$\begin{array}{r|l} 4-28x^2+9x^4 & 2+4x-3x^2 \\ \underline{4+8x-6x^2} & \overline{2-4x-3x^2} \\ -8x-22x^2+9x^4 & \\ \underline{-8x-16x^2+12x^3} & \\ -6x^2-12x^3+9x^4 & \\ \underline{-6x^2-12x^3+9x^4} & \end{array}$$

EXERCISE 23

Divide the following:

1. $15x^2-11x-14$ by $3x+2$.

2. $12a^2-32a+21$ by $6a-7$.

3. $32t^2+28st-15s^2$ by $4t+5s$.

4. $c^3-8c^2-5c+84$ by $c-7$.

5. $a^2-2ab+b^2$ by $a-b$.

6. $a^2+2ab+b^2$ by $a+b$.

7. x^2+4x+4 by $x+2$.

8. x^2-6x+9 by $x-3$.

Note the form of examples 5 to 8, also the results obtained. Have you similar forms in Exercise 18?

9. $k^2-8k+15$ by $k-3$

10. h^2-h-12 by $h-4$.

11. a^3-1 by $a-1$.

12. a^3-8b^3 by $a-2b$.

Have you had multiplication problems similar to examples 9 to 12?

13. $64z^3+27d^3$ by $4z+3d$.

14. $8(b+x)^3-y^3$ by $2(b+x)-y$.

15. $x^3-5x^2y+9xy^2-9y^3$ by $x^2-2xy+3y^2$.

16. n^4-16 by $n+2$.

17. a^5+243 by $a+3$.

Do the quotients in examples 11, 12, 13, 14, 16, 17 seem to have similarity of form?

18. $16(a-b)^2-9$ by $4(a-b)+3$.

19. $\frac{8}{9}a^2-\frac{1}{3}a-\frac{3}{4}$ by $\frac{2}{3}a+\frac{1}{2}$.

20. $\frac{1}{27}g^3-\frac{1}{12}g^2k+\frac{1}{16}gk^2-\frac{1}{64}k^3$ by $\frac{1}{3}g-\frac{1}{4}k$.

21. e^4-81 by $e^3-3\,e^2+9\,e-27$.

22. $a^4-256\,b^4$ by $a-4\,b$. Compare example 16.

23. $13\,x^3+6\,x^4-70\,x^2+71\,x-20$ by $4+3\,x^2-7\,x$.

24. $42(c+d)^3-47(c+d)^2+17(c+d)-2$ by $7(c+d)-2$.

25. $m^5-18\,m^3-3\,m^4+24\,m^2+52\,m-21$ by $m+m^2-7$.

26. $\frac{8}{27}\,x^3+\frac{27}{64}$ by $\frac{2}{3}\,x+\frac{3}{4}$.

27. $9\,h^4-52\,h^2k^2+64\,k^4$ by $3\,h^2-2\,hk-8\,k^2$.

28. $6\,x^5+5\,x^4-57\,x^3-x^2+67\,x+28$ by $-4+3\,x^2-5\,x$.

29. $a^{2x}-b^{2n}+2\,b^nc^z-c^{2z}$ by $a^x+b^n-c^z$.

30. $4\,a^{2m+7}b^3-11\,a^{m+6}b^{x+1}+6\,a^5b^{2x-1}$ by $a^{m+2}b-2\,ab^{x-1}$.

67. The operation of division is often facilitated by the use of parentheses.

Ex. Divide $x^3+(a+b-c)x^2+(ab-bc-ca)x-abc$ by $x+a$.

$$
\begin{array}{l|l}
x^3+(a+b-c)x^2+(ab-bc-ca)x-abc & x+a \\
\cline{2-2}
\underline{x^3+\qquad\quad ax^2} & x^2+(b-c)x-bc \\
\qquad\quad (b-c)x^2 & \\
\qquad\quad \underline{(b-c)x^2+(ab\qquad -ca)x} & \\
\qquad\qquad\qquad\qquad\quad -bcx & \\
\qquad\qquad\qquad\qquad\quad \underline{-bcx-abc} &
\end{array}
$$

EXERCISE 24

Divide the following:

1. $x^3+(a-b-c)x^2+(-ab+bc-ca)x+abc$
 by $x^2+(a-b)x-ab$.

2. $x^3+(a+b-c)x^2+(ab-bc-ca)x-abc$ by $x-c$.

3. $x^3-(a+b+c)x^2+(ab+bc+ca)x-abc$ by $x^2-(b+c)x+bc$.

4. $x^3-(a-2\,b-3\,c)x^2+(-2\,ab+6\,bc-3\,ca)x-6\,abc$
 by $x^2-(a-3\,c)x-3\,ac$.

5. $x^3+(3\,a+b+2\,c)x^2+(3\,ab+2\,bc+6\,ca)x+6\,abc$
 by $x+3\,a$.

6. $m(m+n)x^2-(m^2+n^2)x+n(m-n)$ by $mx-n$.

7. $x^2-(m-2n)x-2m^2+11mn-15n^2$ by $x+m-3n$.

8. $(2m^2+10mn)x^2+(8m^2-9mn-15n^2)x-(12mn-9n^2)$ by $2mx-3n$.

9. $x^3-(3a+2b-4c)x^2+(6ab-8bc+12ca)x-24abc$ by $x-2b$.

10. $a(a-b)x^2+(-ab+b^2+bc)x-c(b+c)$ by $(a-b)x+c$.

QUERIES

1. Is the continued product of six numbers, one half of which are positive and one half negative, a positive or a negative number? Why?

2. What three numbers are involved in these two problems:

$$(a+b)(a+b);\ (a^2+2ab+b^2)\div(a+b)?$$

3. Make a rule governing the result of $(a+b)(a+b)$.

4. Will the rule in 3 govern the result of $(x+7)(x+7)$?

5. Apply your rule to example 10, Exercise 18.

6. Find the product of $(a+b)$ and $(a-b)$ and make a general rule for such a product.

7. Can you solve example 8, Exercise 18, by this rule?

8. One of two factors of 30 is 6, what is the other? How did you find it?

9. One of two factors of x^2-x-12 is $x-4$, what is the other? Does this correspond to your definition of division?

10. Does your rule in 3 aid you in solving such problems as example 6, Exercise 23? Such forms are of frequent occurrence.

11. What is the quotient of $a^2+12a+36$ divided by $a+6$?

12. Given the multiplicand $=m$, and the product $=p$, what is the multiplier?

13. The product is z^4+z^2+1, the multiplier is z^2-z+1; find the multiplicand.

VI. INTEGRAL LINEAR EQUATIONS

68. Any term of either member of an equation (§ 3) is called a *term* of the equation.

69. A **Numerical Equation** is one in which all the known numbers are represented by Arabic numerals; as,

$$2x-7=x+6.$$

An **Integral Equation** is one each of whose members is a rational and integral expression (§ 57); as,

$$4x-5=\frac{2}{3}y+1.$$

70. An **Identical Equation**, or **Identity**, is an equation which is always true for specified values of the letters which enter; as,
$$(a+b)(a-b)\equiv a^2-b^2.$$

The sign $\equiv$, read "*is identically equal to*," is frequently used in place of the sign of equality in an identity.

71. An equation is said to be *satisfied* by a set of values of certain letters involved in it when, on substituting the value of each letter in place of the letter wherever it occurs, the equation becomes identical.

Thus, the equation $x-y=5$ is satisfied by the set of values $x=8$, $y=3$; for, on substituting 8 for x, and 3 for y, the equation becomes $8-3=5$, or $5=5$; which is identical.

72. An **Equation of Condition** is an equation involving one or more letters, called **Unknown Numbers**, which is satisfied only by particular values of these letters.

Thus, the equation $x+2=5$ is not satisfied by every value of x, but only by the particular value $x=3$.

An equation of condition is usually called an *equation*.

Any letter in an equation of condition may represent an unknown number.

73. If an equation contains but one unknown number, any value of the unknown number which satisfies the equation is called a **Root** of the equation.

Thus, 3 is a root of the equation $x+2=5$,

To *solve* an equation is to find its roots.

74. If a rational and integral monomial (§ 57) involves a certain letter, its *degree with respect to it* is denoted by its exponent (§ 58).

If it involves two letters, its *degree with respect to them* is denoted by the sum of their exponents; etc.

Thus, $2\,ab^4x^2y^3$ is of the *second* degree with respect to x, and of the *fifth* with respect to x and y.

75. If an integral equation (§ 69) contains one or more unknown numbers, the *degree* of the equation is the degree of its term of highest degree.

Thus, if x and y represent unknown numbers,

$ax-by=c$ is an equation of the *first* degree;

$x^2+4\,x=-2$, an equation of the *second* degree;

$2\,x^2-3\,xy^2=5$, an equation of the *third* degree; etc.

A **Linear**, or **Simple, Equation** is an equation of the first degree.

PRINCIPLES USED IN SOLVING INTEGRAL EQUATIONS

76. Since the members of an equation are equal numbers, we may write the last four axioms of § 4 as follows:

1. The same number, or equal numbers, may be added to both members of an equation without destroying the equality.

2. The same number, or equal numbers, may be subtracted from both members of an equation without destroying the equality.

3. Both members of an equation may be multiplied by the same number, or equal numbers, without destroying the equality.

4. Both members of an equation may be divided by the same number, or equal numbers, without destroying the equality.

77. Transposing Terms. — Consider the equation

$$x+a-b=c.$$

Adding $-a$ and $+b$ to both members (§ 76, 1), we have

$$x=c-a+b.$$

In this case, the terms $+a$ and $-b$ are said to be *transposed* from the first member to the second.

Similarly, any term may be transposed from one member of an equation to the other by changing its sign.

78. It follows from § 77 that

If the same term occurs in both members of an equation affected with the same sign, it may be cancelled.

79. Consider the equation

$$a-x=b-c. \qquad (1)$$

Multiplying each term by -1 (§ 76), we have

$$x-a=c-b;$$

which is the same as equation (1) with the sign of every term changed.

Similarly, the signs of all the terms of an equation may be changed, without destroying the equality.

80. Clearing of Fractions.—Consider the equation

$$\frac{2}{3}x-\frac{5}{4}=\frac{5}{6}x-\frac{9}{8}.$$

Multiplying each term by 24, the lowest common multiple of the denominators (Ax. 7, § 4), we have

$$16x-30=20x-27,$$

where the denominators have been removed.

Removing the fractions from an equation by multiplication is called *clearing the equation of fractions*.

SOLUTION OF INTEGRAL LINEAR EQUATIONS

81. To solve an equation involving one unknown number, we put it into a succession of forms, which finally leads to the value of the root.

This process is called *transforming* the equation.

Every transformation is effected by means of the principles of §§ 76 to 80, inclusive.

82. Examples.

1. Solve the equation $5x-7=3x+1$.

Transposing $3x$ to the first member, and -7 to the second (§ 77), we have

$$5x-3x=7+1.$$

Uniting similar terms, $2x=8$.

Dividing both members by 2 (§ 76, 4),

$$x=4.$$

To *verify* the result, put $x=4$ in the given equation.

Thus, $20-7=12+1$; which is identical.

2. Solve the equation

$$\frac{7}{6}t-\frac{5}{3}=\frac{3}{5}t-\frac{1}{4}.$$

Clearing of fractions by multiplying each term by 60, the L. C. M. of 6, 3, 5, and 4, we have

$$70t-100=36t-15.$$

Transposing $36t$ to the first member, and -100 to the second,

$$70t-36t=100-15.$$

Uniting terms, $34t=85$.

Dividing by 34, $t=\frac{85}{34}=\frac{5}{2}$.

Verify this result by substituting $t=\frac{5}{2}$ in the given equation.

3. Solve the equation

$$(5-3x)(3+4x)=62-(7-3x)(1-4x).$$

Expanding, $15+11x-12x^2=62-(7-31x+12x^2)$.

Or, $15+11x-12x^2=62-7+31x-12x^2$.

Cancelling the $-12x^2$ terms (§ 78), and transposing,

$$11x-31x=62-7-15.$$

Uniting terms, $-20x=40$.

Dividing by -20, $x=-2$.

Verify the result by substituting $x=-2$ in the given equation.

To *expand* an algebraic expression is to perform the operations indicated.

From these examples, we have the following rule for solving an integral linear equation with one unknown number:

Clear the equation of fractions, if any, by multiplying each term by the L. C. M. of the denominators of the fractional terms.

Remove the parentheses, if any, by performing all the operations indicated.

Transpose the unknown terms to the first member, and the known to the second; cancelling any term which has the same coefficient in both members.

Unite similar terms, and divide both members by the coefficient of the unknown number.

The pupil should verify every result.

EXERCISE 25

Solve the following equations, in each case verifying the result:

1. $5x+13=28$.
2. $7z=4z-33$.
3. $11n+71=6n+76$.
4. $8d-2=5d-26$.
5. $15x+19=11x-5$.
6. $13-21k=34-14k$.
7. $25x-3=4+18x$.
8. $24t-28=14t-48$.
9. $26-4x=31-2x$.
10. $16R-47=8R-43$.
11. $17+14x=11x+16$.
12. $43k-27=37-149k$.
13. $12y+15=15y+17$.
14. $98-16x=23-41x$.
15. $29x-8+17=32x-14x-24$.
16. $35z-41=-81+63z-58z$.
17. $0=31t-14t+3t+30$.
18. $\frac{1}{3}m+\frac{1}{2}=2-\frac{5}{6}-\frac{1}{2}m$.
19. $\frac{2}{3}v-\frac{1}{6}=\frac{5}{6}v+\frac{1}{3}$.

20. $\frac{5}{6}k+\frac{1}{6}k+\frac{8}{3}k=\frac{11}{3}$.

21. $\frac{5}{6}s=\frac{7}{4}s-\frac{3}{8}s-\frac{13}{48}$.

22. $\frac{4}{7}q+\frac{5}{4}=\frac{9}{14}q+\frac{3}{28}q$.

23. $\frac{2}{5}x-\frac{38}{3}=\frac{8}{9}x-\frac{4}{3}x$.

24. $\frac{2}{3}v-\frac{4}{5}v=\frac{7}{8}v-\frac{1}{20}v-\frac{23}{24}$.

25. $5(x+3)-7=6(2x-3)+40$.

26. $12k-(4k-7)=3k-(9k-28)$.

27. $75-8(7u+5)=6u-(4u+52)$.

28. $5R-3(2-8R)=9R-4(1-4R)$.

29. $(4-3z)(5+4z)=(8+2z)(1-6z)-82$.

30. $\frac{1}{3}(4x+1)+\frac{1}{5}(6x-2)-\frac{1}{6}(5x+8)=2$.

PROBLEMS LEADING TO INTEGRAL LINEAR EQUATIONS WITH ONE UNKNOWN NUMBER

83. For the solution of a problem by algebraic methods, the following suggestions will be found of service:

1. Represent the unknown number, or one of the unknown numbers if there are several, by *some* letter, as x.

2. Every problem contains, explicitly or implicitly, *at least as many distinct statements as there are unknown numbers involved.* Use all but one of these to express the other unknown numbers in terms of x.

3. Use the remaining statement to form an equation.

84. Problems.

1. Divide 45 into two parts such that the less part shall be one-fourth the greater.

Here there are *two* unknown numbers; the greater part and the less.

In accordance with the first suggestion of § 83, we represent the greater part by x.

The first statement of the problem is, implicitly:

The sum of the greater part and the less is 45.

The second statement is:

The less part is one-fourth the greater.

In accordance with the second suggestion of § 83, we use the *first statement* to express the less part in terms of x.

Thus, the less part is represented by $45-x$.

We now, in accordance with the third suggestion, use the *second statement* to form an equation.

Thus, $45-x=\frac{1}{4}x.$

Clearing of fractions, $180-4x=x.$

Transposing, $-4x-x=-180$, or $-5x=-180.$

Dividing by -5, $x=36$, the greater part.

Then, $45-x=9$, the less part.

Verify by substituting $x=36$ in the given equation.

2. A had twice as much money as B; but after giving B \$28, he has $\frac{2}{3}$ as much as B. How much had each at first?

Let x represent the number of dollars B had at first.

Then, $2x$ will represent the number A had at first.

Now after giving B \$28, A has $2x-28$ dollars, and B, $x+28$ dollars; we then have the equation

$$2x-28=\frac{2}{3}(x+28).$$

Clearing of fractions, $6x-84=2(x+28).$

Expanding, $6x-84=2x+56.$

Transposing, $4x=140.$

Dividing by 4, $x=35$, the number of dollars B had at first;

and $2x=70$, the number of dollars A had at first.

Verify the result.

3. A is 3 times as old as B, and 8 years ago he was 7 times as old as B. Required their ages at present.

Let $n=$ the number of years in B's age.

Then, $3n=$ the number of years in A's age.

Also, $n-8=$ the number of years in B's age 8 years ago,

and $3n-8=$ the number of years in A's age 8 years ago.

But A's age 8 years ago was 7 times B's age 8 years ago.

Whence, $3n-8=7(n-8).$

Expanding, $3n-8=7n-56.$

Transposing, $-4n=-48.$

Dividing by -4, $n=12$, the number of years in B's age.

Whence, $3n=36$, the number of years in A's age.

Verify the result.

4. A sum of money amounting to \$4.32 consists of 108 coins, all dimes and cents; how many are there of each kind?

Let $\qquad x =$ the number of dimes.

Then, $\qquad 108 - x =$ the number of cents.

Also, the x dimes are worth $10x$ cents.
But the entire sum amounts to 432 *cents*.

Whence, $\qquad 10x + 108 - x = 432.$

Transposing, $\qquad 9x = 324.$

Whence, $\qquad x = 36$, the number of dimes;

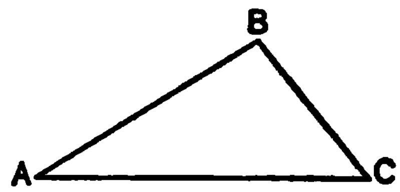
and $\qquad 108 - x = 72$, the number of cents.

Verify the result.

EXERCISE 26

1. The difference of two numbers is 12, and 7 times the smaller exceeds the greater by 30. Find the numbers.

2. The sum of two sides, AB and BC, of the triangle ABC is 23, and the lesser side exceeds their difference by 7. Find the sides AB and BC. Can more than one such triangle be drawn?

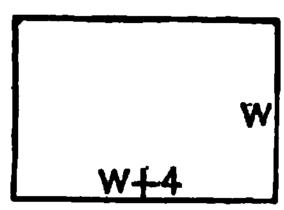


3. Find two numbers whose sum is $\frac{7}{6}$, and difference $\frac{1}{6}$.

4. The sum of two numbers is 35, and their difference is three-fifths the larger number. Find the numbers.

5. A is 5 years older than B, and the sum of their ages is 39 years. How old is each?

6. A rectangle is 4 feet longer than it is wide. If 4 feet were added to the length the area would be increased 40 square feet. Find the length of the sides.



7. A rectangle is 6 feet longer than it is wide. If 3 feet be added to its width and 4 feet be subtracted from its length its area will not be changed. Find the length and breadth.

8. A man counting the coins he has in his hand finds that he has three times as many quarters as half dollars, five

more dimes than quarters, and twice as many five-cent pieces as dimes. The entire sum of money is \$2.85. How many coins of each kind?

9. A is 25 years of age, B is 9 years of age. In how many years will A be twice as old as B?

10. The length of a rectangle is 5 feet more than the width. If 4 feet be taken from the length and 4 feet from the width the area of the rectangle will be diminished 124 square feet. Find the length and breadth of the rectangle.

11. A certain number of two digits is equal to 9 times the sum of the digits and the digit in ten's place is 7 greater than the digit in unit's place. Find the number.

12. Divide \$300 among A, B, and C so that $\frac{1}{3}$ of B's share plus \$20 may equal A's share, and C and B may have equal amounts.

13. A man has \$4.10, all five-cent and fifty-cent pieces; and he has 5 more five-cent pieces than fifty-cent pieces. How many has he of each?

14. The difference between $\frac{3}{4}$ and $\frac{1}{3}$ of a certain number exceeds $\frac{1}{9}$ of it by 44. What is the number?

15. A has \$5.50 and B \$3.50; how much money must A give B in order that B may have $\frac{4}{5}$ as much as A?

16. A room is $\frac{8}{5}$ as long as it is wide; if the length were diminished 3 feet and the width increased by the same amount, the room would be square. Find its dimensions.

Note: Oranges come packed in boxes, a box containing 86, 90, 110, 126, 150, 175 oranges. A box marked 90's indicates that there are 90 oranges in that box.

17. A merchant buys oranges, 150's, a certain number of boxes at \$3.25, twice as many at \$3.00 and six boxes at \$3.50, paying \$39.50 for the entire lot. Find the average cost per dozen oranges.

18. A merchant buys oranges, 90's, some at \$2.00 per box, $\frac{2}{5}$ as many at \$2.20 per box, paying \$28.80 for the entire lot. Can he make a profit retailing them at 29¢ per dozen, no allowance being made for expense of handling?

Note: Banana dealers estimate the value of a *bunch* of bananas by the number of *hands* on a bunch. A *hand* is a cluster of bananas grouped together and contains 12 to 16 bananas.

19. A merchant bought three lots of bananas; some, 8 hands, at 85¢; three times as many 12 hands at \$1.15, and 5 bunches, 10 hands, at \$1.05, paying \$18.15 for all. Find the approximate average cost per dozen bananas, averaging 15 bananas to the hand.

20. A given square has 39 square feet more area than a given rectangle. The length of the rectangle is 3 feet more than a side of the square, and the breadth of the rectangle is 5 feet less than a side of the square. Find the dimensions of each figure.

21. Divide \$480 among A, B, C, and D so that B shall have twice as much as A, B shall have \$6 more than C, and C and D together as much as A and B together.

22. Find two numbers whose difference is 17, such that the square of the greater exceeds the square of the less by 1037.

23. A room is $\frac{3}{2}$ as long as it is wide, and 60 feet of picture molding are required to go around it. Find the number of square feet in the floor.

24. A starts to walk from Boston to Rockland, 19 miles, at the same time, B starts to walk from Rockland to Boston. A walks $\frac{1}{2}$ mile an hour faster than B. They meet in $3\frac{1}{2}$ hours. Find the rate of each.

(Let R = number of miles per hour A walks.)

25. The sum of \$900 is invested, part at 4%, and the rest at 5%, per annum, and the total annual income is \$42. How much is invested in each way?

26. In 9 years B will be $\frac{5}{6}$ as old as A; and 12 years ago he was $\frac{3}{5}$ as old. What are their ages?

(Let n represent the number of years in A's age 12 years ago.)

27. A man buys irrigated farm land, some at \$17 per acre, and three times as much less 160 acres at \$15 per acre, paying \$17,440 for the entire farm. He also pays \$2.50 an acre for a water right. He sells the land for \$21 per acre. What is his profit?

28. A has $\frac{1}{5}$ of a certain sum of money, B has $\frac{2}{7}$, C \$5 less than $\frac{3}{8}$, D the balance which is \$44. Find C's share.

29. Find three consecutive numbers such that the square of the greatest exceeds the product of the other two by 70.

30. Find three consecutive numbers such that if the square of the least number be subtracted from the product of the other two the remainder will be 47.

31. A number consists of two digits, and the ten's digit is 5 greater than the unit's digit. The difference between the squares of the digits is 65. What is the number?

32. A is 10 years older than B; 4 years ago B was $\frac{2}{3}$ as old as A will be in 5 years. Find the age of each.

33. There are two heaps of coins, the first containing 5-cent pieces, and the second 10-cent pieces. The second heap is worth 20 cents more than the first, and has 8 fewer coins. Find the number in each heap.

34. A certain number is composed of two digits; the number is six more than six times the sum of the digits, and the digit in unit's place is $\frac{2}{3}$ the digit in ten's place. Find the number.

35. Find four consecutive odd numbers such that the product of the first and third shall be less than the product of the second and fourth by 64.

VII. PRODUCTS AND FACTORS

85. A Power of a Power. — Required the value of $(a^2)^3$.

By § 6, $(a^2)^3 = a^2 \times a^2 \times a^2 = a^6$.

The general case: — Required the value of $(a^m)^n$, where m and n are any positive integers.

We have, $(a^m)^n = a^m \times a^m \times \cdots$ to n factors

$$= a^{m+m+\cdots \text{ to } n \text{ terms}} = a^{mn}.$$

86. A Power of a Product. — Required the value of $(ab)^3$.

By § 6, $(ab)^3 = ab \times ab \times ab = a^3b^3$.

The general case: — Required the value of $(ab)^n$, where n is any positive integer.

We have, $(ab)^n = ab \times ab \times \cdots$ to n factors $= a^nb^n$.

In like manner, $(abc\cdots)^n = a^nb^nc^n\cdots$,

whatever the number of factors in $abc\cdots$.

87. A Power of a Monomial.

1. Find the value of $(-5a^4)^3$.

By § 26, $(-5a^4)^3 = [(-5)\times a^4]^3$

$$= (-5)^3 \times (a^4)^3 \text{ (§ 86)} = -125\,a^{12} \text{ (§ 85)}.$$

2. Find the value of $(-2m^3n)^4$.

We have, $(-2m^3n)^4 = (-2)^4 \times (m^3)^4 \times n^4 = 16\,m^{12}n^4$.

88. From §§ 85 and 86 and the examples of § 87, we have the following rule for raising a rational and integral monomial (§ 57) to any power whose exponent is a positive integer.

Raise the absolute value of the numerical coefficient to the required power, and multiply the exponent of each letter by the exponent of the required power.

Give to every power of a positive term, and to every *even* power of a negative term, the positive sign; and to every *odd* power of a negative term the negative sign.

EXERCISE 27

Expand the following:

1. $(x^3y^4z^5)^3$.
2. $(m^6n^2p)^{11}$.
3. $(-ab^7c^{10})^7$.
4. $(-11\,x^9y^8)^2$.
5. $(7\,a^mb^{2n})^3$.
6. $(-n^3x^5y^4)^{10}$.
7. $(2\,m^6x^7)^6$.
8. $(-4\,x^2yz^{11})^4$.
9. $(a^nb^3c)^p$.
10. $(x^{5m}y^{4n}z^{3p})^{12}$.
11. $(-3\,m^2n^9x^6)^5$.
12. $(-2\,am^5n^7)^9$.

Find the factors of the following:

13. $25\,a^2b^4$.
14. $32\,m^5$.
15. $48\,a^4b^2c$.
16. $21\cdot a^7$.
17. $3\,a^3$.
18. a^{2n}.
19. a^{n+1}.
20. a^{2n+1}.
21. $343\,a^3c^4$.
22. $243\,m^3z^6$.
23. $165\,a^4z^2$.
24. $-282\,a^{2m}c^{2x}$.

89. Type I. Product of the Sum and Difference of Two Numbers. — Let it be required to multiply $a+b$ by $a-b$.

$$\begin{array}{r}
a+b \\
\underline{a-b} \\
a^2+ab \\
\underline{-ab-b^2}
\end{array}$$

Whence, $(a+b)(a-b)=a^2 \quad -b^2$.

That is, the product of the sum and difference of two numbers equals the difference of their squares.

1. Multiply $6\,a+5\,b^3$ by $6\,a-5\,b^3$.

By the rule,

$$(6\,a+5\,b^3)(6\,a-5\,b^3)=(6\,a)^2-(5\,b^3)^2=36\,a^2-25\,b^6.$$

2. Multiply $-x^2+4$ by $-x^2-4$.

$$\begin{aligned}(-x^2+4)(-x^2-4)&=[(-x^2)+4][(-x^2)-4]\\ &=(-x^2)^2-4^2=x^4-16.\end{aligned}$$

EXERCISE 28

Expand the following:

1. $(4\,a+3\,b)(4\,a-3\,b)$.
2. $(2\,x+4\,y)(2\,x-4\,y)$.
3. $(3\,c+8)(3\,c-8)$.
4. $(8\,R+1)(8\,R-1)$.

5. $(6d+5t)(6d-5t)$.

6. $(11k^4+3l^3)(11k^4-3l^3)$.

7. $(9g+2)(9g-2)$.

8. $(15a+13b)(15a-13b)$.

9. $(-x+7)(x+7)$.

(Prove this last result by actual multiplication.)

10. $(12x+y)(12x-y)$.

11. $(-19c^2+4d^4)(-19c^2-4d^4)$.

12. From what factors do you obtain x^2-9?

13. From what do you obtain $4a^2-25$?

14. Find the factors of $9c^2-49d^2$.

By reversing the product rule in § 89, this rule follows:

To factor the difference of two squares, extract the square root of the first square, and of the second square; add the results for one factor, and subtract the second result from the first for the other factor.

Note: It is not always possible to factor an expression; there are, however, certain forms which can always be factored; these will be considered in the present work.

Factor the following:

(*Check:* If results are correct, the product of the factors will equal the given expression.)

15. $9a^2-4b^2$.

16. $36m^2-25k^2$.

17. a^2-9c^2.

18. $25c^2-81h^2$.

19. $49-4d^2$.

20. $36x^4-121y^6$.

21. $16-25a^8$.

22. $100m^4x^{12}-169y^{10}$.

23. $(m+n)^2-z^2$.

24. $49a^8-144b^6$.

25. $(x+y)^2-(a+b)^2$.

26. $(x+y)^2-(a-b)^2$.

27. $(2a+x)^2-(a-2x)^2$.

Expand the following:

28. $(5a^2+12b^3c)(5a^2-12b^3c)$.

29. $(a^x+e^y)(a^x-e^y)$.

30. $(6m+4b^2)(6m-4b^2)$.

31. $(c^{3x}+d^{3y})(c^{3x}-d^{3y})$.

Sometimes the factors of an expression admit of further factoring:

32. $x^4-81=(x^2+9)(x^2-9)$ (The second factor can be factored)
$=(x^2+9)(x+3)(x-3)$.

33. Factor $16\,m^4-625\,y^4$.

34. Factor a^8-b^8.

35. Factor $81\,c^8-16\,d^4$.

36. Factor $(R+S+3)^2-(R-S-4)^2$.

37. Does $36\,c^2-2$ belong to this type?

38. Can you factor x^4+9 by this type?

90. By division:

$$\frac{a^2-b^2}{a+b}=a-b, \quad (1) \qquad \frac{a^2-b^2}{a-b}=a+b. \quad (2)$$

(Compare Exercise 23, Ex. 18.)

1. Divide $25\,y^2z^4-9$ by $5\,yz^2-3$.

By § 88, $25\,y^2z^4$ is the square of $5\,yz^2$; then by (2),

$$\frac{25\,y^2z^4-9}{5\,yz^2-3}=5\,yz^2+3.$$

2. Divide $x^2-(y-z)^2$ by $x+(y-z)$.

By (1), $$\frac{x^2-(y-z)^2}{x+(y-z)}=x-(y-z)=x-y+z.$$

EXERCISE 29

Find, without actual division, the values of the following:

1. $\dfrac{a^2-4}{a+2}$.

2. $\dfrac{x^2-9}{x-3}$.

3. $\dfrac{25\,n^4-1}{5\,n^2-1}$.

4. $\dfrac{64\,n^8-x^{10}}{8\,n^4+x^5}$.

5. $\dfrac{1-144\,a^{2x}b^{2c}}{1-12\,a^xb^c}$.

6. $\dfrac{49\,x^4z^2-64}{7\,x^2z+8}$.

91. Type II. Square of a Binomial. — Let it be required to square $a+b$.

$$\begin{array}{r} a+b \\ \underline{a+b} \\ a^2+\ ab \qquad \\ \underline{\qquad ab+b^2} \end{array}$$

Whence, $$(a+b)^2=a^2+2\,ab+b^2. \qquad (1)$$

That is, the square of the sum of two numbers equals the square of the first, plus twice the product of the first by the second, plus the square of the second.

1. Square $3a+2b$.

We have, $(3a+2b)^2=(3a)^2+2(3a)(2b)+(2b)^2$
$=9a^2+12ab+4b^2.$

Let it be required to square $a-b$.

$$\begin{array}{r} a-b \\ a-b \\ \hline a^2-\ ab \quad\ \\ -\ ab+b^2 \\ \hline \end{array}$$

Whence, $\quad (\boldsymbol{a-b})^2=\boldsymbol{a^2-2ab+b^2}.$ (2)

That is, the square of the difference of two numbers equals the square of the first, minus twice the product of the first by the second, plus the square of the second.

In the remainder of the work we shall use the expression "the *difference* between a and b" to denote the *remainder obtained by subtracting b from a*.

The result (2) may also be derived by substituting $-b$ for b, in equation (1).

2. Square $4x^2-5$.

We have, $(4x^2-5)^2=(4x^2)^2-2(4x^2)(5)+5^2$
$=16x^4-40x^2+25.$

If the first term of the binomial is *negative*, it should be written, negative sign and all, in parenthesis, before applying the rules.

3. Square $-2a^3+9$.

We have, $(-2a^3+9)^2=[(-2a^3)+9]^2$
$=(-2a^3)^2+2(-2a^3)(9)+9^2$
$=4a^6-36a^3+81.$

EXERCISE 30

The following 18 examples are for mental drill:

1. $(x+3)^2$.
2. $(a-4)^2$.
3. $(c+9)^2$.
4. $(2x+7)^2$.
5. $(4y-6z)^2$.
6. $(3ac-4b)^2$.
7. $(x+4)^2$.
8. $(-4k+3d)^2$.
9. $(h-11)^2$.
10. $(v-12w)^2$
11. $(4a+13b)^2$.
12. $(15x-1)^2$.

Note that in each of these trinomial squares, the first and third terms are perfect squares and positive, and the middle term is twice the product of the square roots of the first and third terms.

What sign does the middle term have?

In each of the following expressions supply the missing term which will form a perfect trinomial square:

13. x^2+4x.
15. c^2+16.
17. b^2-4b.
14. a^2+9.
16. x^2+12x.
18. s^2+4.

Can you substitute other numbers than those you used and still form a perfect square?

19. $x^2+10x+25$ is the square of what?

20. x^2-6x+9 is composed of what factors?

(Compare example 1.)

21. Factor $x^2+2xy+y^2$.

22. From what two factors do you obtain $16a^2+8a+1$?

By reversing the product rule in § 91, this rule follows:

To factor a trinomial square, extract the square roots of the first and third terms, and connect the results by the sign of the second term. This gives one of the equal factors.

23. Factor $4x^2+12xy+9y^2$.
26. Factor $25k^2+60hk+36h^2$.
24. Factor $9y^2+6y+1$.
27. Expand $(3x+2y)^2$.
25. Factor $c^2+8c+16$.
28. Expand $(8x^5+9x^6)^2$.

29. From what do you obtain $x^2y^2+14xy+49$?

Sometimes the factors of an expression admit of further factoring:

30. $x^4-8x^2+16=(x^2-4)(x^2-4)$

$$=(x+2)(x-2)(x+2)(x-2) \quad \text{[by § 89]}.$$

This result may be written $(x+2)^2(x-2)^2$.

31. Factor a^4-18a^2+81.

32. Factor $49t^2+168tu+144u^2$.

33. Factor $25(a+b)^2+40(a+b)c+16c^2$.

34. Factor $16\,m^4-72\,m^2v^2+81\,v^4$.

35. Expand $(x+y+z)(x-y+z)$.

$$\begin{aligned}(x+y+z)(x-y+z)&=[(x+z)+y]\,[(x+z)-y]\\&=(x+z)^2-y^2\\&=x^2+2\,xz+z^2-y^2.\end{aligned}$$

36. Expand $(a+b-c)(a-b+c)$.

By § 46, $(a+b-c)(a-b+c)=[a+(b-c)]\,[a-(b-c)]$
$=a^2-(b-c)^2$, by the rule,
$=a^2-(b^2-2\,bc+c^2)$
$=a^2-b^2+2\,bc-c^2$.

37. $(x^2+x+1)(x^2+x-1)$.

38. $(a^2+1+3\,a)(a^2+1-3\,a)$.

39. $(x+y+3)(x-y-3)$.

40. $(a^2+5\,a-4)(a^2-5\,a+4)$.

41. Factor $a^2+2\,ab+b^2-c^2$.

$$\begin{aligned}&=(a+b)^2-c^2\\&=(a+b+c)(a+b-c).\end{aligned}$$

42. Factor $a^2+6\,a+9-4\,c^2$.

43. Factor $9-a^2+2\,ab-b^2$ (§ 46).

44. Factor $a^2+2\,ab+b^2-c^2-2\,cd-d^2$.

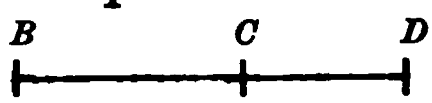
45. Factor $a^2-4\,ax+4\,x^2-b^2+6\,by-9\,y^2$.

46. Factor $x^2-y^2-2\,yz-z^2$.

47. Is $x^2-8\,x+25$ a perfect square? Why?

48. Square both members of the equation

$(BC)=(BD)-(CD)$. (See figure.)



92. Type III. Product of Two Binomials having the Same First Term. — Let it be required to multiply $x+a$ by $x+b$.

$$\begin{array}{l}x+a\\ \underline{x+b\qquad\qquad\quad}\\ x^2+\quad ax\\ \underline{\quad\;\;+\quad bx+ab}\end{array}$$

Whence, $(x+a)(x+b)=x^2+(a+b)x+ab.$

That is, the product of two binomials having the same first term equals the square of the first term, plus the algebraic sum of the second terms multiplied by the first term, plus the product of the second terms.

1. Multiply $x-5$ by $x+3$.

The coefficient of x is the sum of -5 and $+3$, or -2.
The last term is the product of -5 and $+3$, or -15.

Whence, $$(x-5)(x+3)=x^2-2x-15.$$

2. Multiply $x-5$ by $x-3$.

The coefficient of x is the sum of -5 and -3, or -8.
The last term is the product of -5 and -3, or 15.

Whence, $$(x-5)(x-3)=x^2-8x+15.$$

3. Multiply $ab-4$ by $ab+7$.

The coefficient of ab is the sum of -4 and 7, or 3.
The last term is the product of -4 and 7, or -28.

Whence, $$(ab-4)(ab+7)=a^2b^2+3ab-28.$$

4. Multiply x^2+6y^3 by x^2+8y^3.

The coefficient of x^2 is the sum of $6y^3$ and $8y^3$, or $14y^3$.
The last term is the product of $6y^3$ and $8y^3$, or $48y^6$.

Whence, $$(x^2+6y^3)(x^2+8y^3)=x^4+14x^2y^3+48y^6.$$

EXERCISE 31

Expand the following by inspection:

1. $(x+2)(x+3)$.
2. $(x-3)(x+7)$.
3. $(x-12)(x-1)$.
4. $(x-9)(x+2)$.
5. $(z^2+13)(z^2+2)$.
6. $(a^3-1)(a^3+27)$.
7. $(c^5-4)(c^5+6)$
8. $(a^x+3)(a^x+9)$.
9. $(R+2C)(R+9C)$.
10. $(e-8y)(e-9y)$.
11. $(a^3+2)(a^3-5)$.
12. $(x^n-1)(x^n+7)$.
13. $(x^2-20)(x^2+4)$.
14. $(e^x+3)(e^x-11)$.
15. From what factors do you obtain $x^2+8x+15$?
(Compare example 9, Exercise 23.)
16. What are the factors of $x^2+7x+12$?

17. Factor $x^2+4x-12$.

By the rule in § 92, the product takes the form

$$x^2+ax+b.$$

To factor a trinomial of the form

$$x^2+ax+b,$$

reverse this process.

Hence, to obtain the second terms of the binomials reverse the rule for products and find two numbers whose algebraic sum is the coefficient of x, and whose product is the last term of the trinomial. The numbers may be found by inspection.

18. Factor $x^2+14x+45$.

We find two numbers whose sum is 14 and product 45.

By inspection, we determine that these numbers are 9 and 5.

Whence, $x^2+14x+45=(x+9)(x+5)$.

Factor the following:

19. $x^2+3x-10$.
20. $x^2-12x+11$.
21. $x^2-5x-14$.
22. a^4+16a^2+15.
23. $m^2+5m-24$.
24. c^2-c-72.
25. $d^2+37d+36$.
26. k^4+5k^2-14.
27. R^6-13R^3+22.
28. $m^2+6m-16$.
29. $1+2a-99a^2$.
30. $a^2+18a+56$.
31. $c^2-10c-75$.
32. $k^2-6k-72$.
33. $m^2+27m+72$.
34. $a^2+17a+72$.
35. $(x-y)^2-9(x-y)+20$.
36. $(a+b)^2+(a+b)-56$.
37. $(c+d)^2-4(c+d)-60$.

Expand by inspection:

38. $(a^2-8)(a^2+12)$.
39. $(c+7)(c+7)$.
40. $(h+3)(h+3)$.
41. $[(x+y)+2][(x+y)-14]$.
42. $[(m+R)-8]\,[(m+R)+6]$.

Find numbers which will make the following factorable:

43. $x^2+(\ ?)x+36$. 44. $a^2\ (\ )a-72$. 45. $c^2\ (\ )c-48$.

EXERCISE 32

Select the type to which each of the following belongs and then factor:

1. x^4-4x^2-32.
2. $a^2+8a+16$.
3. $a^2+17a+16$.
4. $a^2+10a+16$.
5. $k^2-12k+36$.
6. x^2+2x+1.
7. $36x^2-9y^2$.
8. $a^2-16+2ab+b^2$.
9. a^4-625.
10. $z^2-z-132$.
11. $m^2-50m+49$.
12. $m^2-14m+49$.
13. Can you factor x^2+x+1 by any type you have had?

The accuracy of your factors can always be proved by finding the product of your factors.

14. Factor $(x+y)^2-11(x+y)+30$.
15. Factor $x^2+(2m+3k)x+6mk$.

93. Type IV. Product of Two Binomials of the Form $mx+n$ and $px+q$. — We find by multiplication:

$$\begin{array}{lr}
mx+n & \\
\times & \\
px+q & \\
\hline
mpx^2+ & npx \\
+ & mqx+nq \\
\hline
mpx^2+(np+mq)x+nq &
\end{array}$$

The first term of this result, mpx^2, is the product of the first terms of the binomial factors, and the last term, nq, the product of the second terms.

The middle term, $(np+mq)x$, is the sum of the products of the terms, in the binomial factors, connected by cross lines.

Ex. Multiply $3x+4$ by $2x-5$.

The first term is the product of $3x$ and $2x$, or $6x^2$.

The coefficient of x is the sum of 4×2 and $3\times(-5)$; that is, $8-15$, or -7.

The last term is the product of 4 and -5, or -20.

Whence, $(3x+4)(2x-5)=6x^2-7x-20$.

EXERCISE 33

Expand the following by inspection:

1. $(x+2)(4x+3)$.
2. $(3x-2)(2x+1)$.
3. $(2x-7)(5x+3)$.
4. $(8x-1)(7x+2)$.
5. $(a-6)(3a-4)$.
6. $(2k+15)(4k-11)$.
7. $(6e-5)(4e-3)$.
8. $(2d-1)(5d+2)$.
9. $(3m+4x)(2m-3x)$.
10. $(2a^x+3y)(3a^x+5y)$.
11. $(6a^2+x^2)(8a^2-5x^2)$.
12. $(5R-4H)(3R+11H)$
13. $(m+11b)(11m+b)$.
14. $(6k-5l)(5k+6l)$.

94. Note that the product of two factors of the above form is a trinomial of the form

(Type IV.) $\qquad ax^2+bx+c.$

To factor a trinomial of the form

$$ax^2+bx+c,$$

reverse the above process. Hence,

To resolve a trinomial of the form ax^2+bx+c into two binomial factors, the first terms of the binomials must be such that their product is ax^2; the second terms must be such that their product is c; the sum of the cross products must be bx.

1. Factor $3x^2+8x+4$.

The first terms of the binomial factors must be such that their product is $3x^2$; they can be only $3x$ and x.

The second terms must be such that their product is 4.

The numbers whose product is 4 are 4 and 1, -4 and -1, 2 and 2, and -2 and -2; the possible cases are represented below:

$$\begin{array}{c} x+4 \\ \times \\ \underline{3x+1} \\ 13x \end{array} \qquad \begin{array}{c} x+1 \\ \times \\ \underline{3x+4} \\ 7x \end{array} \qquad \begin{array}{c} x-4 \\ \times \\ \underline{3x-1} \\ -13x \end{array}$$

$$\begin{array}{c} x-1 \\ \times \\ \underline{3x-4} \\ -7x \end{array} \qquad \begin{array}{c} x+2 \\ \times \\ \underline{3x+2} \\ 8x \end{array} \qquad \begin{array}{c} x-2 \\ \times \\ \underline{3x-2} \\ -8x \end{array}$$

The corresponding middle term of the trinomial, obtained by cross-multiplication, as in § 93, is given in each case; and only the factors $x+2$, $3x+2$ satisfy the condition that the middle term shall be $8x$.

Then, $$3x^2+8x+4=(x+2)(3x+2).$$

2. Factor $6x^2-x-2$.

The first terms of the factors must be $6x$ and x, or $3x$ and $2x$.
The second terms must be 2 and -1, or -2 and 1.
The possible cases are given below:

$$\begin{array}{cccc} 6x+2 & 6x-1 & 6x-2 & 6x+1 \\ \times & \times & \times & \times \\ \underline{x-1} & \underline{x+2} & \underline{x+1} & \underline{x-2} \\ -4x & 11x & 4x & -11x \\ \\ 3x+2 & 3x-1 & 3x-2 & 3x+1 \\ \times & \times & \times & \times \\ \underline{2x-1} & \underline{2x+2} & \underline{2x+1} & \underline{2x-2} \\ x & 4x & -x & -4x \end{array}$$

Only the factors $3x-2$ and $2x+1$ satisfy the condition that the middle term shall be $-x$.

Then, $$6x^2-x-2=(3x-2)(2x+1).$$

The following suggestions will be found of service:

(*a*) **If the last term of the trinomial is positive, the last terms of the factors will be both +, or both −, according as the middle term of the trinomial is + or −.**

Thus, in Ex. 1, we need not have tried the numbers -1 and -4, nor -2 and -2; this would have left only three cases to consider.

(*b*) **If the last term of the trinomial is negative, the last terms of the factors will be one +, the other −.**

If the x^2 term is negative, the entire expression should be enclosed in parentheses preceded by a − sign.

If the coefficient of x^2 is a perfect square, and the coefficient of x divisible by the square root of the coefficient of x^2, the expression may be readily factored by the method of § 91.

3. Factor $9x^2-18x+5$.

In this case, 18 is divisible by the square root of 9.

We have $$9x^2-18x+5=(3x)^2-6(3x)+5.$$

We find two numbers whose sum is -6, and product 5.

The numbers are -5 and -1.

Then, $$9x^2-18x+5=(3x-5)(3x-1).$$

EXERCISE 34

Factor the following by inspection:

1. $3x^2+20x+12$.
2. $14x^2+5x-1$.
3. $8x^2-14x-15$.
4. $20a^2-27a+9$.
5. $16m^2+16m+3$.
6. $15R^2+4R-4$.
7. $22a^4-19a^2+4$.
8. $30c^8+41c^4+6$.
9. $10a^2x^2-3ax-18$.
10. $30x^2+17dx-2d^2$.
11. $36x^2-19xy-6y^2$.
12. $49a^2-42ab+8b^2$.
13. $54a^{2x}+15a^xy+y^2$.
14. $48a^4-22a^2x^2-5x^4$.
15. $50t^2-55st+14s^2$.
16. $72c^2d^2-13abcd-15a^2b^2$.

EXERCISE 35

Select the type to which each of the following belongs and then factor:

1. $9b^2-20bc+4c^2$.
2. $9b^2-12bc+4c^2$.
3. $9b^2-4c^2$.
4. $(235)^2-(234)^2$.
5. $9b^2-16bc-4c^2$.
6. $k^2-13ky-48y^2$.
7. $k^2+14ky+49y^2$.
8. $15c^2-19cd-56d^2$.
9. $6x^2-7x-20$.
10. $a^2-16ab+64b^2$.
11. $36d^2x^2-36dmx+9m^2$.
12. $256a^4-800a^2b^2+625b^4$.
13. x^8-y^8.
14. Can you factor $3x^2-2x+12$?

Solve the following equations and verify each result:

15. $(x+3)^2+(x+5)(3x-4)=(2x+5)^2$.
16. $(3t+5)(3t-5)-(t+7)(t-1)=(8t+3)(t-1)$.
17. $(2m-3)^2+(m+8)(m-8)=(5m-1)(m+3)$.

95. It is not possible to factor every expression of the form x^2+ax+b by the method of § 92.

Thus, let it be required to factor $x^2+18x+35$.

We must find two numbers whose sum is 18, and product 35.

The only pairs of positive integral factors of 35 are 7 and 5, and 35 and 1; and in neither case is the sum 18.

It is also impossible to factor every expression of the form ax^2+bx+c by the method of § 94.

Thus, it is impossible to find two binomial factors of the expression $4x^2+4x-1$ by the method of § 94.

In § 236 will be given a *general* method for the factoring of any expression of the form x^2+ax+b, or ax^2+bx+c.

96. Type V. When the expression is in the form

$$x^4+ax^2y^2+y^4.$$

Certain trinomials of the above form may be factored by expressing them as the difference of two perfect squares, and then employing § 89.

1. Factor $a^4+a^2b^2+b^4$.

By § 91, a trinomial is a perfect square if its first and last terms are perfect squares and positive, and its second term plus or minus twice the product of their square roots.

The given expression can be made a perfect square by adding a^2b^2 to its second term; and this can be done provided we subtract a^2b^2 from the result.

Thus, $a^4+a^2b^2+b^4=(a^4+2a^2b^2+b^4)-a^2b^2$
$=(a^2+b^2)^2-a^2b^2$, by § 91,
$=(a^2+b^2+ab)(a^2+b^2-ab)$, by § 89,
$=(a^2+ab+b^2)(a^2-ab+b^2)$.

2. Factor $9x^4-37x^2+4$.

The expression will be a perfect square if its second term is $-12x^2$.

Thus, $9x^4-37x^2+4=(9x^4-12x^2+4)-25x^2$
$=(3x^2-2)^2-(5x)^2$
$=(3x^2+5x-2)(3x^2-5x-2)$.

The expression may also be factored as follows:

$9x^4-37x^2+4=(9x^4+12x^2+4)-49x^2$
$=(3x^2+2)^2-(7x)^2=(3x^2+7x+2)(3x^2-7x+2)$.

Several expressions in Exercise 36 may be factored in two different ways.

The factoring of trinomials of the form $x^4+ax^2y^2+y^4$, when the factors involve surds, will be considered in § 237.

EXERCISE 36

Factor the following:

1. x^4+5x^2+9.
2. $a^4-21a^2b^2+36b^4$.
3. $4-33x^2+4x^4$.
4. $25m^4-14m^2n^2+n^4$.
5. $9x^4+6x^2y^2+49y^4$.
6. $16a^4-81a^2+16$.
7. $64-64m^2+25m^4$.
8. $49a^4-127a^2x^2+81x^4$.

Factor each of the following in two different ways (compare §§ 92, 94):

9. x^4-17x^2+16.
10. $9-148a^2+64a^4$.
11. $16m^4-104m^2x^2+25x^4$.
12. $36a^4-97a^2m^2+36m^4$.

97. Type VI. We find by division,

$$\frac{a^3+b^3}{a+b}=a^2-ab+b^2. \qquad \frac{a^3-b^3}{a-b}=a^2+ab+b^2.$$

That is,

If the sum of the cubes of two numbers be divided by the sum of the numbers, the quotient is the square of the first number, minus the product of the first by the second, plus the square of the second number.

If the difference of the cubes of two numbers be divided by the difference of the numbers, the quotient is the square of the first number, plus the product of the first by the second, plus the square of the second number.

If an expression can be resolved into three equal factors, it is said to be a *perfect cube*, and one of the equal factors is called its *cube root*.

Thus, since $27a^6b^3$ is equal to $3a^2b \times 3a^2b \times 3a^2b$, it is a perfect cube, and $3a^2b$ is its cube root.

Similarly to extract the cube root of a positive monomial perfect cube:

Extract the cube root of the numerical coefficient, and divide the exponent of each letter by 3.

Thus, the cube root of $125\,a^6b^9c^3$ is $5\,a^2b^3c$.

1. Divide $1+8\,a^3$ by $1+2\,a$.

By § 88, $8\,a^3$ is the cube of $2\,a$; then, by the first rule,

$$\frac{1+8\,a^3}{1+2\,a}=\frac{1+(2\,a)^3}{1+2\,a}=1-2\,a+(2\,a)^2=1-2\,a+4\,a^2.$$

(Compare Exs. 11–14, 26, Exercise 23.)

2. Divide $27\,x^6-64\,y^9$ by $3\,x^2-4\,y^3$.

By the second rule,

$$\frac{27\,x^6-64\,y^9}{3\,x^2-4\,y^3}=\frac{(3\,x^2)^3-(4\,y^3)^3}{3\,x^2-4\,y^3}=(3\,x^2)^2+(3\,x^2)(4\,y^3)+(4\,y^3)^2$$
$$=9\,x^4+12\,x^2y^3+16\,y^6.$$

EXERCISE 37

Find, without actual division, the values of the following:

1. $\dfrac{x^3+1}{x+1}$.

2. $\dfrac{1-a^3}{1-a}$.

3. $\dfrac{n^3-27}{n-3}$.

4. $\dfrac{a^6+b^6}{a^2+b^2}$.

5. $\dfrac{a^3+125}{a+5}$.

6. $\dfrac{64\,x^{6m}-1}{4\,x^{2m}-1}$.

7. $\dfrac{27\,x^6-125\,y^3}{3\,x^2-5\,y}$.

8. $\dfrac{343\,m^3n^3+8\,p^3}{7\,mn+2\,p}$.

9. $\dfrac{64\,a^6b^3+216\,c^9}{4\,a^2b+6\,c^3}$.

Factor the following:

10. a^3+b^3.

11. x^3-y^3.

12. $1+m^3n^3$.

13. $8\,a^3+27\,c^3$.

14. $1-27\,n^3$.

15. a^6+1.

16. $64\,m^3-n^3$.

17. $a^3b^3-216\,c^3$.

18. $8\,m^{3p}+27\,n^{3q}$.

98. Type VII. We find by actual division,

$$\frac{a^4-b^4}{a+b}=a^3-a^2b+ab^2-b^3.$$

$$\frac{a^4-b^4}{a-b}=a^3+a^2b+ab^2+b^3.$$

$$\frac{a^5+b^5}{a+b}=a^4-a^3b+a^2b^2-ab^3+b^4.$$

$$\frac{a^5-b^5}{a-b}=a^4+a^3b+a^2b^2+ab^3+b^4;\ \text{etc.}$$

In these results, we observe the following laws:

I. The exponent of a in the first term of the quotient is less by 1 than its exponent in the dividend, and decreases by 1 in each succeeding term.

II. The exponent of b in the second term of the quotient is 1, and increases by 1 in each succeeding term.

III. If the divisor is $a-b$, all the terms of the quotient are positive; if the divisor is $a+b$, the terms of the quotient are alternately positive and negative.

(Compare Exs. 14, 16, 17, Exercise 23.)

1. Divide a^7-b^7 by $a-b$.

By the above laws,

$$\frac{a^7-b^7}{a-b}=a^6+a^5b+a^4b^2+a^3b^3+a^2b^4+ab^5+b^6.$$

2. Divide $16x^4-81$ by $2x+3$.

We have
$$\frac{16x^4-81}{2x+3}=\frac{(2x)^4-3^4}{2x+3}$$
$$=(2x)^3-(2x)^2\cdot 3+2x\cdot 3^2-3^3$$
$$=8x^3-12x^2+18x-27.$$

EXERCISE 38

Find, without actual division, the values of the following:

1. $\dfrac{h^4-k^4}{h-k}$.
2. $\dfrac{m^5+n^5}{m+n}$.
3. $\dfrac{x^6-1}{x-1}$.
4. $\dfrac{1-a^6}{1+a}$.
5. $\dfrac{x^8-y^8}{x^2+y^2}$.
6. $\dfrac{a^{15}-b^5c^{10}}{a^3-bc^2}$.

Factor the following:

7. x^5+y^5.
8. a^5-1.
9. $1-m^5n^5$.
10. $1+x^7$.
11. a^7-b^7.
12. a^9-1.
13. x^9+n^9.
14. $32a^5-b^5$.
15. $n^{5p}+32$.
16. $243x^5+y^5$.
17. $m^{14}+128n^7$.
18. $32a^5b^{15n}-243c^{10p}$.

99. The following statements will be found helpful if n is a positive integer:

$x+y$ is a factor of x^n+y^n if n is *odd*.
$x-y$ is *never* a factor of x^n+y^n.
$x-y$ is *always* a factor of x^n-y^n.
$x+y$ is a factor of x^n-y^n if n is *even*.

When one factor is $x-y$ all the terms of the other factor are *positive*, and when one factor is $x+y$ the terms of the other factor are *alternately positive* and *negative*.

100. A Common Factor of two or more expressions is an expression which is a factor of each of them.

101. Type VIII. When the terms of the expression have a common factor.

1. Factor $14\,ab^4-35\,a^3b^2$.

Each term contains the monomial factor $7\,ab^2$.
Dividing the expression by $7\,ab^2$, we have $2\,b^2-5\,a^2$.

Then, $$14\,ab^4-35\,a^3b^2=7\,ab^2(2\,b^2-5\,a^2).$$

2. Factor $(2\,m+3)x^2+(2\,m+3)y^3$.

The terms have the common binomial factor $2\,m+3$.
Dividing the expression by $2\,m+3$, we have x^2+y^3.

$$2\,m+3\overline{)(2\,m+3)x^2+(2\,m+3)y^3}$$
$$x^2 \qquad +y^3$$

Then, $$(2\,m+3)x^2+(2\,m+3)y^3=(2\,m+3)(x^2+y^3).$$

(See example 6, Exercise 22.)

3. Factor $(a-b)m+(b-a)n$.

By § 46, $$b-a=-(a-b).$$
Then, $$(a-b)m+(b-a)n=(a-b)m-(a-b)n$$
$$=(a-b)(m-n).$$

We may also solve Ex. 3 as follows:

$$(a-b)m+(b-a)n=(b-a)n-(b-a)m=(b-a)(n-m).$$

4. Factor $5\,a(x-y)-3\,a(x+y)$.

$$5\,a(x-y)-3\,a(x+y)=a[5(x-y)-3(x+y)]$$
$$=a(5\,x-5\,y-3\,x-3\,y)$$
$$=a(2\,x-8\,y)=2\,a(x-4\,y).$$

After a common factor is removed one or both of the factors may admit of further factoring.

5. Factor $a^2x^2+2\,a^2xy+a^2y^2$.

Dividing by the common factor a^2, we have for the factors a^2 and $x^2+2\,xy+y^2$.

The trinomial is factorable by § 91.

Whence,
$$a^2x^2+2\,a^2xy+a^2y^2=a^2(x+y)(x+y)$$
$$=a^2(x+y)^2.$$

EXERCISE 39

Factor the following:

1. $36\,m^2-48\,m^2k^2$.
2. $a^5-3\,a^4b+3\,a^3b^2-a^2b^3$.
3. $21\,x^2y-33\,xy^2+12\,xy$.
4. $14\,z^2xc-28\,z^3x^2+7\,z^4xc^2$.
5. $(a+2)c^4-(a+2)d^3a$.
6. $(2\,x+7)x^2+(2\,x+7)$.
7. $(h-k)a^2-(k-h)4\,c^2$.
8. $c^2(c^2-2)+4\,y^2(2-c^2)$.
9. $(x+y)^2+4\,k(x+y)$.
10. $4\,d^3(d-1)-(1-d)$.
11. $4(3\,x+2)+4(2\,x+3)$.
12. $(a-x)^3-5(a-x)^2$.
13. $(2\,m+3)a^2-(2\,m+3)b^2$.
14. $(m-1)a^4-(m-1)b^4$.
15. $(a+b)a^2+(a+b)2\,ab+(a+b)b^2$.
16. $x^2(x^2-1)-8\,x(x^2-1)+15(x^2-1)$.
17. $(m-d)^4-2\,m(m-d)^3+m^2(m-d)^2$.

In every expression to be factored first remove the common factor, if any, then factor the remaining part if possible.

Sometimes it is necessary to group the terms (§§ 46, 47), to show a common factor, then apply the method of Type VIII.

18. $ab-ay+bx-xy$.

By § 46, $\qquad ab-ay+bx-xy=a(b-y)+x(b-y)$.

The terms now have the common factor $b-y$.

Whence, $\qquad ab-ay+bx-xy=(b-y)(a+x)$.

19. $a^3+2\,a^2-3\,a-6$.

If the third term is negative it is convenient to write the last two terms in parenthesis preceded by a $-$ sign, § 46.

Thus,
$$a^3+2\,a^2-3\,a-6=(a^3+2\,a^2)-(3\,a+6)$$
$$=a^2(a+2)-3(a+2)$$
$$=(a+2)(a^2-3).$$

20. $ac+ad+bc+bd$.

21. $xy-3x+2y-6$.

22. $mx+my-nx-ny$.

23. $ab-a-5b+5$.

24. $8xy+12ay+10bx+15ab$.

25. $m^4+6m^3-7m-42$.

26. $6-10a+27a^2-45a^3$.

27. $20ab-28ad-5bc+7cd$.

Be sure that the factors of your final result will not admit of further factoring.

28. $x^3+2x^2y+xy^2$.

29. $a+6ab+9ab^2$.

30. $x^2(a+b)-49y^2(a+b)$.

31. $108k^2s^3-36ks^3+3s^3$.

32. $m^2(2m+3)-3m(2m+3)-10(2m+3)$.

33. $9t^2(3t+2)+8t(3t+2)+4(3t+2)$.

34. $d^3(d+3c)+27c^3(d+3c)$.

35. $5a^2x^2-10a^2xy+5a^2y^2-20a^2z^2$.

36. $48a^6-243a^2b^4$.

Solve the following by inspection:

37. $98^2=(100-2)^2$

$=(10000-400+4)$

$=9604$.

38. $99^2=?$

39. $104^2=?$

40. $35^2=?$

41. $65^2=?$

42. $98^2-2^2=?$

43. $102^2-98^2=?$

44. $68^2=?$

45. $78^2=?$

46. $76^2-4^2=?$

47. $97^2-93^2=?$

48. $111^2-11^2=?$

The examples under Type IV afford a valuable application of the method in Type VIII.

49. Factor $6x^2-7x-20$.

Multiply -20 by 6 (the coefficient of x^2). Factor -120 so that the sum of the factors is -7 (the coefficient of x). These factors are -15, 8.

Then write $6x^2-7x-20=6x^2-15x+8x-20$.

Group by Type VIII, $=3x(2x-5)+4(2x-5)$,

whence, $6x^2-7x-20=(2x-5)(3x+4)$.

50. Factor examples 1–10, Exercise 34, by this method.

102. Hints on Factoring.

For *all expressions:*

First, try Type VIII.

Sometimes the common factor is disguised as in examples 8 and 19, Exercise 39.

Second, select the *type form* to which the expression belongs:

Test *binomials* by means of Types I, VI, VII.

Sometimes the binomial form is disguised. See examples 27 and 28, Exercise 40.

Test *trinomials* by means of Types II, III, IV, V.

Third, be sure that no factor in the result will admit of further factoring.

TYPE FORMS

I. $a^2-b^2=(a+b)(a-b)$. (§ 89)

II. $a^2+2ab+b^2=(a+b)(a+b)$,
$a^2-2ab+b^2=(a-b)(a-b)$. (§ 91)

III. x^2+ax+b. (§ 92)

IV. ax^2+bx+c. (§ 94)

V. $x^4+ax^2y^2+y^4$. (§ 96)

VI. $a^3+b^3=(a+b)(a^2-ab+b^2)$,
$a^3-b^3=(a-b)(a^2+ab+b^2)$. (§ 97)

VII. a^n-b^n,
a^n+b^n. (§ 98)

VIII. $ax+ay+az=a(x+y+z)$. (§ 101)

MISCELLANEOUS AND REVIEW EXAMPLES

EXERCISE 40

Factor the following:

1. $42a^2bc-7ab$.
2. $x^2-5x-36$.
3. $3a(a-x)+3a(c+d)$.
4. $36d^2-72dR+35R^2$.

5. a^4-64.

6. $v+\frac{1}{273}vt$.

7. $8a^2-14ab-15b^2$.

8. $27x^3-8z^3$.

9. $vt+\frac{1}{2}gt^2$.

10. c^4-d^4.

11. $125a^4-50a^3b+5a^2b^2$.

12. $a(b+c)-a(b-c)$.

13. $x^2(5y-2z)-x^2(2y+z)$.

14. $a^6-16a^3c^3+64c^6$.

15. a^6-26a^3-27.

16. $x^{14}-2x^7+1$.

17. $ax-ay+az-bx+by-bz$.

18. $(a+b)^2+14(a+b)+24$.

19. $(x-y)^2-15(x-y)-16$.

20. $4c^2(c+d)+12cd(c+d)+9d^2(c+d)$.

21. $2c^2(2c+3d)+5cd(2c+3d)+3d^2(2c+3d)$.

22. $18x^2-27abx-35a^2b^2$.

23. $x^2+(5c+2d)x+10cd$.

24. $7x^2(3a-2b)-3x^2(2a-3b)$.

After factors are found always unite any similar terms which occur in parenthesis.

25. $(x^2+x-2)^2-(x^2-x+3)^2$.

26. $64a^3+1000$.

27. $a^2-c^2-d^2+b^2-2ab-2cd$.

28. $36m^2-(x-y)^2+12m+1$.

29. $3(m+n)^2-2(m^2-n^2)$.

30. $2a^7x-8a^5x^3+2a^3x^5-8ax^7$.

31. $2xy-2x^2y^2-264x^3y^3$.

32. $8a(2-3y+x)+5c(3y-x-2)$.

33. h^2-k^2+h+k.

34. m^3+m+x^3+x.

Find the factors common (§ 100) to the following expressions:

35. x^2+x-6, $4x^2-11x+6$.

36. a^2-9c^2, $a^2+4ac-21c^2$, a^3-27c^3.

37. $xy+3cx+2cy+6c^2$, $y^3-5cy^2-24c^2y$.

38. $x(x^2+2x+2)+2(x^2+2x+2)$, x^4+4.

39. $(2c-y)4c^2-(2c-y)4cy+(2c-y)y^2$, $(2c-y)^3$

40. $6a^2+a-2$, $90a^3-25a^2-10a$, $4a^2+2a-2$.

41. x^3+2x^2+2x+1, x^3+1.

42. Solve, using factoring: A square, 441 feet on a side, has a grass plot within it, 432 feet on a side. The remaining part of the square is a concrete walk. Find the cost of the walk at 14¢ per square foot.

Additional work in factoring will be found in §§ 236 and 237.

SOLUTION OF EQUATIONS BY FACTORING

103. The solution of equations affords an important and interesting application of factoring.

Let it be required to solve the equation

$$(x-3)(2x+5)=0.$$

It is evident that the equation will be satisfied when x has such a value that *one* of the factors of the first member is equal to zero; for if any factor of a product is equal to zero, the product is equal to zero.

Hence, the equation will be satisfied when x has such a value that either

$$x-3=0, \tag{1}$$

or

$$2x+5=0. \tag{2}$$

Solving (1) and (2), we have $x=3$ or $-\frac{5}{2}$.

It will be observed that **the roots are obtained by placing the factors of the first member separately equal to zero, and solving the resulting equations.**

104. Examples.

1. Solve the equation $x^2-5x-24=0$.

Factoring the first member, $(x-8)(x+3)=0$. (§ 92)

Placing the factors separately equal to 0 (§ 103), we have

$$x-8=0, \text{ whence } x=8;$$

and

$$x+3=0, \text{ whence } x=-3.$$

Verify by substituting $x=8$, $x=-3$ successively in the given equation.

2. Solve the equation $4x^2-2x=0$.

Factoring the first member, $2x(2x-1)=0$.

Placing the factors separately equal to 0, we have

$$2x=0, \text{ whence } x=0;$$

and $$2x-1=0, \text{ whence } x=\frac{1}{2}.$$

Verify these results.

3. Solve the equation $x^3+4x^2-x-4=0$.

Factoring the first member, we have by §§ 89, 101,

$$(x+4)(x^2-1)=0, \text{ or } (x+4)(x+1)(x-1)=0.$$

Then, $$x+4=0, \text{ whence } x=-4;$$

$$x+1=0, \text{ whence } x=-1;$$

and $$x-1=0, \text{ whence } x=1.$$

Verify these results.

4. Solve the equation $x^3-27-(x^2+9x-36)=0$.

Factoring the first member, we have by §§ 92 and 97,

$$(x-3)(x^2+3x+9)-(x-3)(x+12)=0.$$

Or, $$(x-3)(x^2+3x+9-x-12)=0.$$

Or, $$(x-3)(x^2+2x-3)=0.$$

Or, $$(x-3)(x+3)(x-1)=0.$$

Placing the factors separately equal to 0, $x=3$, -3, or 1. Verify.

The pupil should endeavor to put down the values of x without actually placing the factors equal to 0, as shown in Ex. 4.

EXERCISE 41

Solve each equation and verify results:

1. $x^2-4x-21=0$.
2. $x^2-4x=0$.
3. $6x^3-12x^2=0$.
4. $(2x-7)(x^2-16)=0$.
5. $t^2-t-12=0$.
6. $z^2-8z+12=0$.
7. $k^2+7k+12=0$.
8. $6v^2-17v+12=0$.
9. $9v^2(2v-3)-9v(2v-3)-4(2v-3)=0$.
10. $3x^2-kx-4k^2=0$. (Solve for x, then solve for k.)
11. $10u^2-7u-12=0$.
12. $xz+2x-3z-6=0$.
13. $15v^2+v-2=0$
14. $4x^3+20x^2-9x-45=0$.
15. $28t^2-t-2=0$.
16. $18x^2-27abx-35a^2b^2=0$.

17. $n^2+14\,n-32=0.$

18. $x^2+8\,x+16=0.$

19. $m^3+6\,m^2-9\,m-54=0.$

20. $(x-3)^2-(3\,x+2)^2=0.$

21. $10\,v^2-39\,v+14=0.$

22. $15\,x^2+x-6=0.$

23. $(4\,x^2-49)(x^2-3\,x-10)(8\,x^2+14\,x-15)=0.$

24. $(x-2)(5\,x^2+8\,x-4)-(x^2-4)=0.$

25. What number added to its square gives 30?

26. What number subtracted from 4 times its square gives $\frac{1}{2}$?

27. If to 4 times the square of a certain number we add three times the number the result is 10. Find the number.

28. A rectangular room is 4 feet longer than it is wide, and its area is 96 square feet. What are its dimensions?

Let $w=$ the number of feet in the width,
then $w+4=$ the number of feet in the length.

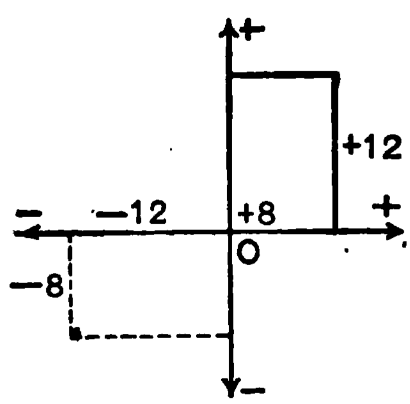
$$w(w+4)=96,$$
$$w^2+4\,w-96=0,$$
$$(w-8)(w+12)=0.$$

Whence, $w=8 \text{ or } -12.$

Then, $w+4=12 \text{ or } -8.$

Since we are finding dimensions of a room, these negative roots have no significance and can be rejected. There is, however, a very interesting geometrical interpretation which may be given. Consider § 10 and Exercise 4. If measurement to the right is positive, then measurement to the left is negative. If distance upward is +, then distance downward is −.

Now draw this rectangle:



This gives two rectangles which fulfill the conditions of the problem, if one remembers that -12 is algebraically less than -8.

29. In a right triangle ABC, the base, AC, is 3 feet more than the altitude, BC, and the area is 14 square feet. Find AC and BC. Make a diagram with your results.

30. The perimeter of a rectangular field is 180 feet, and its area 1800 square feet. Find its dimensions. Make a diagram of your results.

Find the equations whose roots (§ 73) are:

31. $2, -\frac{5}{3}$.

Subtracting each root from x, we have

$$(x-2), \quad (x+\tfrac{5}{3}).$$

By reversing § 103, the product of these expressions equated to zero gives the required equation.

Whence, $(x-2)(x+\frac{5}{3})=0$, or expanding,

$$3x^2-x-10=0.$$

32. $1, 3$.

33. $\frac{2}{5}, \frac{5}{2}$.

34. $-1, 4$.

35. $2, -3, 4$.

36. a, b.

37. $\frac{a}{2}, \frac{a}{4}, a$.

38. $6, -\frac{13}{2}$.

39. $\frac{16}{9}, 0$.

40. $a+2b, a-2b$.

41. The sides of a rectangle are 8 and 11. Form a problem similar to problem 28. State the equation.

QUERIES

1. Is $2a$ a number? Is it a sum? Is it a product? What are its factors?
2. Is $a+b$ a number? Is it the sum of two numbers? Can you factor it?
3. Translate a^2+b^2 into English. Can you factor it?
4. Given two numbers F and S; if their sum is multiplied by their difference, what is the result?
5. Given two numbers F and S; if their sum be multiplied by itself, what is the result? Express in English.
6. Does the *definition of division* bear any relation to your idea of the *process of factoring?*
7. Is $4a^2+2a+1$ a perfect square? Why?
8. The following are for mental drill: $(30\frac{1}{2})^2=?$ $(20\frac{3}{4})^2=?$ $(29\frac{1}{2})^2=?$
9. Is 3 a root of the equation $3x^2-4x+7=0$? Why? Is $x-3$ a factor of the expression?
10. Is 2 a root of $2m^2-9m+10=0$? Is $m-2$ a factor of the expression?
11. How do you form the equation whose roots are 3 and 7?
12. If one root, 5, of $x^2-8x+15=0$ is given, can you find the other root without solving the equation?
13. Using your knowledge of § 91, can you make a general statement covering the results of examples 13 and 14, Exercise 12?

VIII. HIGHEST COMMON FACTOR. LOWEST COMMON MULTIPLE

(We consider in the present chapter the Highest Common Factor and Lowest Common Multiple of *Monomials*, or of *Polynomials* which can be readily factored by inspection.

The Highest Common Factor and Lowest Common Multiple of polynomials which cannot be readily factored by inspection, will be considered in a more advanced course in algebra.)

HIGHEST COMMON FACTOR

105. The **Highest Common Factor** (H. C. F.) of two or more expressions is their common factor of *highest degree* (§ 58).

If several common factors are of equally high degree, it is understood that *the* highest common factor is the one having the numerical coefficient of greatest absolute value in its term of highest degree.

For example, if the common factors were $6x$ and $2x$, the former would be the H. C. F.

106. Two expressions are said to be *prime to each other* when unity is their highest common factor.

107. Case I. **Highest Common Factor of Monomials.**

Ex. Required the H. C. F. of $42a^3b^2$, $70a^2bc$, and $98a^4b^3d^2$.

By the rule of Arithmetic, the H. C. F. of 42, 70, and 98 is 14.

It is evident by inspection that the expression of highest degree which will exactly divide a^3b^2, a^2bc, and $a^4b^3d^2$ is a^2b.

Then, the H. C. F. of the given expressions is $14a^2b$.

It will be observed, in the above result, that *the exponent of each letter is the lowest exponent with which it occurs in any of the given expressions.*

EXERCISE 42

Find the H. C. F. of the following:

1. $14x^3y$, $21xy^4$.
2. $64a^5b^3$, $112b^4c^4$.
3. $36m^3b^2$, $48m^2b^3$, $60m^4b$.
4. $25ac^2$, $30a^2c^2$, $35ac$.

5. $32\,a^4x^4,\ 128\,a^6b^2x^3,\ 192\,a^8x^2y^3.$

6. $136\,a^4m^5n^3,\ 51\,b^3mn^6,\ 119\,c^2m^3n^9.$

7. $60(x-y)^4,\ 84(x-y)^5.$

108. By § 48, $(+a)\times(+b)=+ab,\ (+a)\times(-b)=-ab,$
$(-a)\times(+b)=-ab,\ (-a)\times(-b)=+ab.$

Hence, in the indicated product of two factors, **the signs of both factors may be changed without altering the product; but if the sign of either one be changed, the sign of the product will be changed.**

If either factor is a polynomial, care must be taken, on changing its sign, to change the sign of *each of its terms.*

Thus, $(b-a)(n-m)$ may be written in the form

$$-(b-a)(m-n),\ \text{or}\ -(a-b)(n-m).$$

In like manner, in the indicated product of more than two factors, **the signs of any *even* number of them may be changed without altering the product; but if the signs of any *odd* number of them be changed, the sign of the product will be changed** (§ 49).

Thus, $(a-b)(c-d)(e-f)$ may be written in the forms

$$(a-b)(d-c)(f-e),$$
$$(b-a)(c-d)(f-e),$$
$$-(b-a)(d-c)(f-e),\ \text{etc.}$$

109. Case II. **Highest Common Factor of Polynomials which can be readily factored by Inspection.**

1. Required the H. C. F. of

$$5\,x^4y-45\,x^2y\ \text{and}\ 10\,x^3y^2-40\,x^2y^2-210\,xy^2.$$

By §§ 101, 89, and 92, $5\,x^4y-45\,x^2y=5\,x^2y(x^2-9)$
$=5\,x^2y(x+3)(x-3);$ (1)

and $10\,x^3y^2-40\,x^2y^2-210\,xy^2=10\,xy^2(x^2-4\,x-21)$
$=10\,xy^2(x-7)(x+3).$ (2)

The H. C. F. of the numerical coefficients 5 and 10 is 5.

It is evident by inspection that the H. C. F. of the literal portions of the expressions (1) and (2) is $xy(x+3)$.

Then, the H. C. F. of the given expressions is $5\,xy(x+3)$.

It is sometimes necessary to change the form of the factors in finding the H. C. F. of expressions.

2. Find the H. C. F. of a^2+2a-3 and $1-a^3$.

By § 92, $a^2+2a-3=(a-1)(a+3)$.

By § 97, $1-a^3=(1-a)(1+a+a^2)$.

By § 108, the factors of the first expression can be put in the form

$$-(1-a)(3+a).$$

Hence, the H. C. F. is $1-a$.

EXERCISE 43

Find the H. C. F. of the following:

1. $10x^3y^2-40x^2y^3,\quad 25xy^2-15xy^4$.
2. $c^2-25b^2,\quad c^2-10bc+25b^2$.
3. $a^2-5a-36,\quad a^2-4a-32$.
4. $tz+5z-7t-35,\quad t^2z+8tz+15z$.
5. $2a^2-ab-3b^2,\quad 3a^2+ab-2b^2$.
6. $9a^2-25b^2,\quad 9a^2-30ab+25b^2$.
7. $8n^3+1,\quad 4n^2-2n+1$.
8. $n^2-3n-40,\quad n^2+4n-5,\quad 2n^2+6n-20$.
9. $t^3+2t^2+t+2,\quad t^4+3t^2+2$.
10. $v^3-v^2-v+1,\quad v^4-2v^2+1$.
11. $6a^2+a-2,\ 12a^2+5a-2,\ 6a^2+4a-15az-10z$.
12. $25k^2-16,\quad 25k^2-40k+16,\quad 30k^2+k-20$.
13. $a^5-32,\quad a^2+9a-22,\quad a^3-8$.
14. $1-11a+18a^2,\ 8a^3-1,\ 18a^2-5a-2$.
15. $x^4+3x^2-40,\quad x^4-25,\quad a^3+a^2-5a-5$.
16. $a^2-(b+c)^2,\quad (b-a)^2-c^2,\quad b^2-(a-c)^2$.
17. $(x^2+x+2)(x^2-x-2),\ x^2-5x-6,\ x^2-8x-9$.
18. $2a^2(2a+3t)+5at(2a+3t)+3t^2(2a+3t)$ and $4a^2(t+a)+12at(t+a)+9t^2(t+a)$.

LOWEST COMMON MULTIPLE

110. A **Common Multiple** of two or more expressions is an expression which is exactly divisible by each of them.

111. The **Lowest Common Multiple** (L. C. M.) of two or more expressions is their common multiple of *lowest degree.*

If several common multiples are of equally low degree, it is understood that *the* lowest common multiple is the one having the numerical coefficient of least absolute value in its term of highest degree.

For example, if the common multiples were $4x-2$ and $6x-3$, the former would be the L. C. M.

112. CASE I. **Lowest Common Multiple of Monomials.**

Ex. Required the L. C. M. of $36a^3x$, $60a^2y^2$, and $84cx^3$.

By the rule of Arithmetic, the L. C. M. of 36, 60, and 84 is 1260.

It is evident by inspection that the expression of lowest degree which is exactly divisible by a^3x, a^2y^2, and cx^3 is $a^3cx^3y^2$.

Then, the L. C. M. of the given expressions is $1260a^3cx^3y^2$.

It will be observed, in the above result, that *the exponent of each letter is the highest exponent with which it occurs in any of the given expressions.*

EXERCISE 44

Find the L. C. M. of the following:

1. $5x^3y^3$, $6x^2y^4$.
2. $18a^6b$, $45b^5c$.
3. $28x^5$, $36y^4$.
4. $42m^4n^2$, $98n^5p^3$.
5. $105a^2b$, $70b^2c$, $63c^2a$.
6. $50x^4y^5$, $24x^5y^3$, $40x^6y^4$.
7. $21ab^4$, $35b^2c^6$, $91a^3c^3$.
8. $56a^2b^3$, $84bx^5$, $48x^4y^6$.
9. $60a^3bc^2$, $75a^5b^6d$, $90a^4c^7d^6$.
10. $99m^4nx^8$, $66m^3n^9y^5$, $165n^5x^4y^7$.

113. CASE II. **Lowest Common Multiple of Polynomials which can be readily factored by Inspection.**

1. Required the L. C. M. of

$$x^2-5x+6,\ x^2-4x+4, \text{ and } x^3-9x.$$

By § 92, $x^2-5x+6=(x-3)(x-2).$

By § 91, $x^2-4x+4=(x-2)^2.$

By § 89, $x^3-9x=x(x+3)(x-3).$

It is evident by inspection that the L. C. M. of these expressions is

$$x(x-2)^2(x+3)(x-3).$$

It is sometimes necessary to change the form of the factors.

2. Find the L. C. M. of $ac-bc-ad+bd$ and b^2-a^2.

By § 101, $ac-bc-ad+bd=(a-b)(c-d)$.
By § 89, $b^2-a^2=(b+a)(b-a)$.
By § 108, the factors of the first expression can be written

$$(b-a)(d-c).$$

Hence, the L. C. M. is $(b+a)(b-a)(d-c)$, or $(b^2-a^2)(d-c)$.

EXERCISE 45

Find the L. C. M. of the following:

1. $x^2-7x+10$, $x^2-8x+15$.
2. k^2-4, $k^2-7k+10$, $k^3-5k^2+4k-20$.
3. $2a^2-a-1$, $2a^2+5a+2$, $2a^2+7a+3$.
4. R^2-3R+2, R^2-5R+6, R^2-4R+3.
5. a^2-2a-3, a^2-3a+2, a^2-1.
6. $m-2$, m^2-2m+4, $m^3-6m^2+12m-8$.
7. x^4+x^2+1, x^2+x+1, x^2-x+1.
8. $k+l$, $k+3l$, $l-k$, $k-3l$.
9. $(x-2)(x-3)$, $(x-3)(x-4)$, $(4-x)(2-x)$:
10. a^2-9, a^3-27, $a-3$, a^2+3a+9.

Find the H. C. F. and the L. C. M. of the following:

11. m^2+3m+2, m^2+5m+6, m^2+4x+3.
12. $a^2+4ab+4b^2$, a^2-4b^2, a^2+2ab.
13. $2k^2+7k-4$, $3k^2+13k+4$.
14. $2x^2-3ax+a^2$, $2x^2-5ax+2a^2$, $x^2-3ax+2a^2$.
15. $9t^2-25v^2$, $6tx+10vx$, $12tx+20vx$.

Find the L. C. M. of the following:

16. $9x^3-12x^2+4x$, $18ax^4+12ax^3+8ax^2$, and $27x^3-8$.
17. $(x+z)^2-y^2$, $(x+y)^2-z^2$, $x^2-(y+z)^2$.
18. $(c-1)^2+3c$, c^3-1, $c-1$.
19. $(e+y)^2-4ey$, $e^3+2e^2y+ey^2$, e^4+ey^3.
20. x^3+8, $4x^2-(x^2+4)^2$.

IX. FRACTIONS

114. The quotient of a divided by b is written $\frac{a}{b}$.

The expression $\frac{a}{b}$ is called a **Fraction**; the dividend a is called the *numerator*, and the divisor b the *denominator*.

The numerator and denominator are called the *terms* of the fraction.

115. It follows from § 62, (3), that

If the terms of a fraction be both multiplied, or both divided, by the same expression, the value of the fraction is not changed.

116. By the Rule of Signs in Division (§ 61),

$$\frac{+a}{+b}=\frac{-a}{-b}=-\frac{+a}{-b}=-\frac{-a}{+b}.$$

That is, **if the signs of both terms of a fraction be changed, the sign before the fraction is not changed; but if the sign of either one be changed, the sign before the fraction is changed.**

If either term is a polynomial, care must be taken, on changing its sign, to change the sign of *each of its terms*.

Thus, the fraction $\frac{a-b}{c-d}$, by changing the signs of both numerator and denominator, can be written $\frac{b-a}{d-c}$ (§ 44).

117. It follows from §§ 108 and 116 that **if either term of a fraction is the indicated product of two or more factors, the signs of any *even* number of them may be changed without changing the sign before the fraction: but if the signs of any *odd* number of them be changed, the sign before the fraction is changed.**

Note: To change the sign of a factor is to change the sign of every term of the factor.

Thus, the fraction $\frac{a-b}{(c-d)(e-f)}$ may be written

$$\frac{a-b}{(d-c)(f-e)},\ \frac{b-a}{(d-c)(e-f)},\ -\frac{b-a}{(d-c)(f-e)},\ \text{etc.}$$

EXERCISE 46

Write each of the following in three other ways without changing its value:

1. $\frac{a}{2}$. 2. $\frac{n+3}{7}$. 3. $\frac{8}{2-x}$. 4. $\frac{2x-7}{x+2}$. 5. $\frac{6x-5}{(x-3)(y+4)}$.

6. Write $\frac{(3x-1)(a-4)}{(x+5)(y-2)}$ in four other ways without changing its value.

REDUCTION OF FRACTIONS

118. Reduction of a Fraction to Lower Terms.

A fraction is said to be in its *lowest terms* when its numerator and denominator are prime to each other (§ 106).

(We consider in this text those cases only in which the numerator and denominator can be readily factored by inspection.

The cases in which the numerator and denominator cannot be readily factored by inspection are considered in the second course.)

119. By § 115, dividing both terms of a fraction by the same expression, or cancelling common factors in the numerator and denominator, does not alter the value of the fraction.

We then have the following rule:

Resolve both numerator and denominator into their factors, and cancel all that are common to both.

1. Reduce $\frac{24a^4b^2cx}{40a^2b^2c^2d^3}$ to its lowest terms.

We have, $$\frac{24a^4b^2cx}{40a^2b^2c^2d^3}=\frac{2^3\times3\times a^4b^2cx}{2^3\times5\times a^2b^2c^2d^3}=\frac{3a^2x}{5cd^3},$$

by cancelling the common factor $2^3\times a^2b^2c$.

2. Reduce $\frac{x^3-27}{x^2-2x-3}$ to its lowest terms.

By §§ 97 and 92, $$\frac{x^3-27}{x^2-2x-3}=\frac{(x-3)(x^2+3x+9)}{(x-3)(x+1)}=\frac{x^2+3x+9}{x+1}.$$

3. Reduce $\frac{ax-bx-ay+by}{b^2-a^2}$ to its lowest terms.

By §§ 89 and 101, $\dfrac{ax-bx-ay+by}{b^2-a^2}=\dfrac{(a-b)(x-y)}{(b+a)(b-a)}$.

By § 117, the signs of the terms of the factors of the numerator can be changed without altering the value of the fraction; and in this way the first factor of the numerator becomes the same as the second factor of the denominator.

Then, $\dfrac{ax-bx-ay+by}{b^2-a^2}=\dfrac{(b-a)(y-x)}{(b+a)(b-a)}=\dfrac{y-x}{b+a}$.

If all the factors of the numerator are cancelled, 1 remains to form a numerator; if all the factors of the denominator are cancelled, it is a case of exact division.

EXERCISE 47

Reduce each of the following to its lowest terms:

1. $\dfrac{5\,x^4y^5z^2}{3\,xy^5z^6}$.
2. $\dfrac{12\,a^5b^4}{42\,b^2c^3}$.
3. $\dfrac{54\,mn^2}{99\,m^5n^3}$.
4. $\dfrac{63\,x^3y^4z^7}{84\,x^5y^4z^2}$.
5. $\dfrac{126\,a^6b^3c^5}{14\,a^6c^4}$.
6. $\dfrac{26\,m^3n^4p^2}{130\,m^4n^6p^7}$.
7. $\dfrac{90\,a^3m^7n^4}{36\,am^7n^8}$.
8. $\dfrac{88\,x^4y^6z^3}{66\,x^5yz^9}$.
9. $\dfrac{120\,a^7b^4c^{10}}{75\,ab^9c^2}$.
10. $\dfrac{15\,x^4y+10\,x^3y^2}{6\,x^3y^4+4\,x^2y^5}$.
11. $\dfrac{x^2-9\,x+18}{x^2+x-12}$.
12. $\dfrac{m^2-5\,m-84}{a^2m^2-a^2m-56\,a^2}$.
13. $\dfrac{6\,x^2-7\,xz-20\,z^2}{4\,x^2-25\,z^2}$.
14. $\dfrac{4\,a^2-16\,ad+15\,d^2}{4\,a^2-12\,ad+9\,d^2}$.
15. $\dfrac{a^2+a-12}{3\,a^2-13\,a+12}$.
16. $\dfrac{(x^2-49)(x^2-16x+63)}{(x^2-14\,x+49)(x^2-2\,x-63)}$.
17. $\dfrac{12\,a^4m^2+48\,a^4-10\,m^2b-40\,b}{36\,a^8-60\,a^4b+25\,b^2}$.
18. $\dfrac{36\,a^2+97\,ac+36\,c^2}{9\,a^2+13\,ac+4\,c^2}$.
19. $\dfrac{18\,a^2-3\,ac-10\,c^2}{36\,a^2-25\,c^2}$.
20. $\dfrac{27\,b^3-8\,a^3}{16\,a^2-32\,ab+12\,b^2}$.
21. $\dfrac{165\,t^2+2\,t-1}{15\,t^4+14\,t^3-t^2}$.

22. $\dfrac{(4a-d)^2-(2b-3c)^2}{(4a-2b)^2-(3c-d)^2}$. 23. $\dfrac{(x-y)^2-(2v-u^2)^2}{(x-2v)^2-(u^2-y)^2}$.

24. $\dfrac{4c^2(2c-3d)-6cd(2c-3d)+9d^2(2c-3d)}{10c^2+cd-24d^2}$.

120. Reduction of a Fraction to an Integral or Mixed Expression.

A Mixed Expression is a polynomial consisting of a rational and integral expression (§ 57), with one or more fractions.

Thus, $a+\dfrac{b}{c}$, and $\dfrac{x}{3}+\dfrac{2x-y}{x-y}$ are mixed expressions.

121. A fraction may be reduced to an integral or mixed expression by the operation of division, if the degree (§ 58) of the numerator is equal to, or greater than, that of the denominator.

1. Reduce $\dfrac{6x^2+15x-2}{3x}$ to a mixed expression.

By § 65, $\dfrac{6x^2+15x-2}{3x}=\dfrac{6x^2}{3x}+\dfrac{15x}{3x}-\dfrac{2}{3x}=2x+5-\dfrac{2}{3x}$.

2. Reduce $\dfrac{12x^3-8x^2+4x-5}{4x^2+3}$ to a mixed expression.

$$
\begin{array}{r|l}
4x^2+3)\,12x^3-8x^2+4x-5 & (3x-2 \\
\underline{12x^3\qquad\quad+9x} & \\
-8x^2-5x-5 & \\
\underline{-8x^2\qquad\quad-6} & \\
-5x+1 &
\end{array}
$$

Since the dividend is equal to the product of the divisor and quotient, plus the remainder, we have

$$12x^3-8x^2+4x-5=(4x^2+3)(3x-2)+(-5x+1).$$

Dividing both members by $4x^2+3$, we have

$$\frac{12x^3-8x^2+4x-5}{4x^2+3}=3x-2+\frac{-5x+1}{4x^2+3}.$$

Thus, a remainder of lower degree than the divisor may be written over the divisor in the form of a fraction, and the result added to the quotient.

If the first term of the numerator is negative, as in Ex. 2, it is usual to *change the sign of each term of the numerator*, changing the sign before the fraction (§ 116).

Thus, $\dfrac{12x^3-8x^2+4x-5}{4x^2+3}=3x-2-\dfrac{5x-1}{4x^2+3}.$

EXERCISE 48

Reduce each of the following to a mixed expression:

1. $\dfrac{25a^2-10a+11}{5a}.$
2. $\dfrac{16m^4+12m^3+8m^2-9}{3m^2}.$
3. $\dfrac{4x^4+1}{2x^2+1}.$
4. $\dfrac{x^2+6x+9}{x^2+3x+9}.$
5. $\dfrac{x^3+y^3}{x-y}.$
6. $\dfrac{8a^3-27c^3}{2a+3c}.$
7. $\dfrac{8a^4+12a^2b^2-15b^4}{2a^2+3b^2}.$
8. $\dfrac{a^5+32}{a-2}.$
9. $\dfrac{c^4+d^4}{c+d}.$
10. $\dfrac{24x^3+21x+19}{4x^2-2x+5}.$
11. $\dfrac{x^2-4x+8}{x^2+2x-7}.$
12. $\dfrac{3x^3+7x^2}{3x^2+x-9}.$
13. $\dfrac{8a^2-22ab-21b^2}{2a-7b}.$
14. $\dfrac{49x^2-96xy+27y^2}{7x^2+xy-18y^2}.$

122. It is evident from § 121 that a mixed expression may be result of division. Since the dividend is equal to the product of the divisor and quotient plus the remainder, to reduce a mixed expression to a fraction, **Multiply the integral part by the denominator of the fraction, add the numerator to this result and write the denominator under this sum.**

Note: If a minus sign precedes the fraction, change each sign in the numerator.

Ex. Reduce $2x-3-\dfrac{4x-5}{x+1}$ to a fractional form.

$$2x-3-\frac{4x-5}{x+1}=\frac{(2x-3)(x+1)-(4x-5)}{x+1}$$
$$=\frac{2x^2-x-3-4x+5}{x+1}=\frac{2x^2-5x+2}{x+1}.$$

EXERCISE 49

1. $c-3+\dfrac{4c-3}{2c}$.
2. $5a-1-\dfrac{6a^2-2}{5a}$.
3. $2n+11+\dfrac{3}{6n+2}$.
4. $\dfrac{3x+4y}{3x-4y}+1$.
5. $\dfrac{a^2+2ab+b^2}{4ab}-1$.
6. $\dfrac{(x+y)^2}{(x-y)^2}-1$.
7. $\dfrac{(3c-8d)^2}{9c^2-64d^2}-1$.
8. $x-4a-\dfrac{-4a^2}{x}$.
9. $x+5a-\dfrac{20a}{x-5a}$.
10. $2t-3u-\dfrac{8t^3+27u^3}{4t^2+6tu+9u^2}$.
11. $2a-5b-\dfrac{4a^2-25b^2}{2a-5b}$.
12. $x^2+2xy+y^2+\dfrac{16y^3}{x-2y}$.
13. $\dfrac{9a^2-4ab+b^2}{4a-2b}-(2a-3b)$.
14. $\dfrac{10a^2-29ac+10c^2}{3a-c}+6a+2c$.

123. Reduction of Fractions to their Lowest Common Denominator. — To reduce fractions to their **Lowest Common Denominator** (L. C. D.) is to express them as equivalent fractions, each having for a denominator the L. C. M. of the given denominators.

Let it be required to reduce $\dfrac{4cd}{3a^2b^3}$, $\dfrac{3m}{2ab^2}$, and $\dfrac{5n}{4a^3b}$ to their lowest common denominator.

The L. C. M. of $3a^2b^3$, $2ab^2$, and $4a^3b$ is $12a^3b^3$ (§ 112).

By § 115, if the terms of a fraction be both multiplied by the same expression, the value of the fraction is not changed.

Multiplying both terms of $\dfrac{4cd}{3a^2b^3}$ by $4a$, both terms of $\dfrac{3m}{2ab^2}$ by $6a^2b$, and both terms of $\dfrac{5n}{4a^3b}$ by $3b^2$, we have

$$\frac{16\,acd}{12\,a^3b^3},\ \frac{18\,a^3bm}{12\,a^3b^3},\ \text{and}\ \frac{15\,b^2n}{12\,a^3b^3}.$$

It will be seen that the terms of each fraction are multiplied by an expression, which is obtained by dividing the L. C. D. by the denominator of this fraction.

Whence the following rule:

Find the L. C. M. of the given denominators.

Multiply both terms of each fraction by the quotient obtained by dividing the L. C. D. by the denominator of this fraction.

Before applying the rule, each fraction should be reduced to its lowest terms.

124. *Ex.* Reduce $\frac{4\,a}{a^2-4}$ and $\frac{3\,a}{a^2-5\,a+6}$ to their lowest common denominator.

We have $a^2-4=(a+2)(a-2)$,

and $a^2-5\,a+6=(a-2)(a-3)$.

Then, the L. C. D. is $(a+2)(a-2)(a-3)$. (§ 113)

Dividing the L. C. D. by $(a+2)(a-2)$, the quotient is $a-3$; dividing it by $(a-2)(a-3)$, the quotient is $a+2$.

Then, by the rule, the required fractions are

$$\frac{4\,a(a-3)}{(a+2)(a-2)(a-3)}\ \text{and}\ \frac{3\,a(a+2)}{(a+2)(a-2)(a-3)}.$$

EXERCISE 50

Reduce the following to their lowest common denominator:

1. $\frac{7\,ab}{6},\ \frac{3\,bc}{10},\ \frac{2\,ca}{15}.$
2. $\frac{5}{2\,m^2n},\ \frac{4}{5\,m^3n^3},\ \frac{6}{7\,mn^2}.$
3. $\frac{3\,x+4\,z}{22\,xy^3},\ \frac{6\,x-5\,y}{33\,yz^2}.$
4. $\frac{11\,c^4p}{12\,a^2b},\ \frac{9\,a^5m}{14\,b^4c},\ \frac{8\,b^3n}{21\,c^3a}.$
5. $\frac{4\,a^2}{4\,a^2-9},\ \frac{2}{6\,a^2-9\,a}.$
6. $\frac{1}{m-n},\ \frac{3\,mn}{2(m-n)^2},\ \frac{2\,m^2n^2}{3(m-n)^3}.$
7. $\frac{3\,n}{n^3-8},\ \frac{5}{n^2-4\,n+4}.$
8. $\frac{2}{a^3+3\,a^2+2\,a+6},\ \frac{3\,a}{a^3+27}.$

ADDITION AND SUBTRACTION OF FRACTIONS

125. By § 65, $\frac{b}{a}+\frac{c}{a}-\frac{d}{a}=\frac{b+c-d}{a}$.

We then have the following rule:

To add or subtract fractions, reduce them, if necessary, to equivalent fractions having the lowest common denominator.

Add or subtract the numerator of each resulting fraction, according as the sign before the fraction is + or −, and write the result over the lowest common denominator.

The final result should be reduced to its lowest terms.

126. Examples.

1. Simplify $\frac{4a+3}{4a^2b}+\frac{1-6b^2}{6ab^3}$.

The L. C. D. is $12a^2b^3$; multiplying the terms of the first fraction by $3b^2$, and the terms of the second by $2a$, we have

$$\frac{4a+3}{4a^2b}+\frac{1-6b^2}{6ab^3}=\frac{12ab^2+9b^2}{12a^2b^3}+\frac{2a-12ab^2}{12a^2b^3}$$

$$=\frac{12ab^2+9b^2+2a-12ab^2}{12a^2b^3}=\frac{9b^2+2a}{12a^2b^3}.$$

If a fraction whose numerator is a polynomial is preceded by a − sign, it is convenient to write the numerator in parenthesis preceded by a − sign, as shown in the last term of the numerator in equation (A), of Ex. 2.

If this is not done, care must be taken to *change the sign of each term of the numerator* before combining it with the other numerators.

2. Simplify $\frac{5x-4y}{6}-\frac{7x-2y}{14}$.

The L. C. D. is 42; whence,

$$\frac{5x-4y}{6}-\frac{7x-2y}{14}=\frac{35x-28y}{42}-\frac{21x-6y}{42}$$

$$=\frac{35x-28y-(21x-6y)}{42} \qquad \text{(A)}$$

$$=\frac{35x-28y-21x+6y}{42}=\frac{14x-22y}{42}=\frac{7x-11y}{21}.$$

3. Simplify $\frac{1}{x^2+x}-\frac{1}{x^2-x}$.

We have, $x^2+x=x(x+1)$, and $x^2-x=x(x-1)$.

Then, the L. C. D. is $x(x+1)(x-1)$, or $x(x^2-1)$.

Multiplying the terms of the first fraction by $x-1$, and the terms of the second by $x+1$, we have

$$\frac{1}{x^2+x}-\frac{1}{x^2-x}=\frac{x-1}{x(x^2-1)}-\frac{x+1}{x(x^2-1)}$$

$$=\frac{x-1-(x+1)}{x(x^2-1)}=\frac{x-1-x-1}{x(x^2-1)}=\frac{-2}{x(x^2-1)}.$$

By changing the sign of the numerator, at the same time changing the sign before the fraction (§ 116), we may write the answer $-\frac{2}{x(x^2-1)}$.

Or, by changing the sign of the numerator, and of the factor x^2-1 of the denominator (§ 117), we may write it $\frac{2}{x(1-x^2)}$.

4. Simplify $\frac{1}{a^2-3a+2}-\frac{2}{a^2-4a+3}+\frac{1}{a^2-5a+6}$.

We have, $a^2-3a+2=(a-1)(a-2)$, $a^2-4a+3=(a-1)(a-3)$, and $a^2-5a+6=(a-2)(a-3)$.

Then, the L. C. D. is $(a-1)(a-2)(a-3)$.

Whence, $$\frac{1}{a^2-3a+2}-\frac{2}{a^2-4a+3}+\frac{1}{a^2-5a+6}$$

$$=\frac{a-3}{(a-1)(a-2)(a-3)}-\frac{2(a-2)}{(a-1)(a-2)(a-3)}+\frac{a-1}{(a-1)(a-2)(a-3)}$$

$$=\frac{a-3-2(a-2)+a-1}{(a-1)(a-2)(a-3)}=\frac{a-3-2a+4+a-1}{(a-1)(a-2)(a-3)}$$

$$=\frac{0}{(a-1)(a-2)(a-3)}=0.$$

EXERCISE 51

Simplify the following:

1. $\dfrac{2x+9}{8}+\dfrac{3x-5}{12}.$

2. $\dfrac{5}{4a^3c^2}+\dfrac{8}{7a^5c}.$

3. $\dfrac{2a-3b}{10}-\dfrac{3a-8b}{15}.$

4. $\dfrac{5R+2t}{6R^2t^2}-\dfrac{3R+8t}{9Rt^3}.$

5. $\dfrac{2c-7d^3}{13d^3}-\dfrac{5c+2d}{26c}.$

6. $\dfrac{4a+3b}{2ab}-\dfrac{c+2b}{3bc}+\dfrac{5a-c}{4ac}.$

7. $\dfrac{2(6n+5)}{11}-\dfrac{3(n+6)}{22}+\dfrac{4(5n-4)}{44}.$

8. $\dfrac{2a+3t}{14}-\dfrac{3a+2t}{21}+\dfrac{5a-7t}{28}.$

9. $\dfrac{3a^2-4}{10a^2}-\dfrac{4a^3+2}{5a^3}-\dfrac{6a^4-2}{25a^4}.$

10. $\dfrac{5x-4}{8x}-\dfrac{3y+2}{12y}-\dfrac{2z+5}{6z}.$

11. $\dfrac{5x-7}{5}-\dfrac{9x-8}{15}-\dfrac{12x-11}{20}+\dfrac{2x+9}{10}.$

12. $\dfrac{7t-4}{4}-\dfrac{3t-8}{5}-\dfrac{7t+7}{8}+\dfrac{6t-5}{10}.$

13. $\dfrac{2}{9}(3a+4b)-\dfrac{5}{18}(2a-5b)+\dfrac{7}{12}(a+2b).$

14. $\dfrac{3}{3x-1}+\dfrac{2}{2x+1}.$

15. $\dfrac{1}{m+3}-\dfrac{2}{m-5}.$

16. $\dfrac{5}{m+5}+\dfrac{m}{m-3}.$

17. $\dfrac{x-3}{x-4}-\dfrac{x-4}{x-5}.$

18. $\dfrac{3x}{3x+y}-\dfrac{y}{3x-y}.$

19. $\dfrac{4}{4a-12}+\dfrac{5}{10a+15}.$

20. $\dfrac{5c}{5c-3}+\dfrac{c^2+8c-9}{c^2+4c-2}.$

21. $\dfrac{x}{2x-3}-\dfrac{x}{2x+3}-\dfrac{2x-6}{4x^2-9}.$

22. $\dfrac{3x+2}{3x-2}-\dfrac{9x^2+4}{9x^2-4}.$

23. $\dfrac{6a-5}{a^2-2a-15}+\dfrac{2a-1}{2a^2-5a-25}.$

24. $\dfrac{6a^2-4a}{a^3+27}-\dfrac{a-5}{6a^2+17a-3}.$

25. $\dfrac{5xy+7y^2}{3x-2y}-2y.$

Note: We may regard an integer as a fraction whose denominator is 1.

26. $\frac{2c-1}{3}+4-c-\frac{2c+1}{c+1}$.

27. $\frac{2m-1}{m-1}+\frac{4}{m+1}-\frac{3m}{m^2-1}-2$.

28. $\frac{3z+1}{5z-7}-\frac{2z^2-10z+12}{10z^2-49z+49}$.

29. $\frac{3t}{t+1}-\frac{15}{3t^2+t-2}+5$.

30. $\frac{c+d}{v}+\frac{c^2-d^2}{c^2+cd}-\frac{c}{v}+\frac{d}{c}$.

31. $\frac{3}{c-d}+\frac{2}{c+d}+5+\frac{1}{c+2d}$.

In certain cases, the principles of §§ 116 and 117 enable us to change the form of a fraction to one which is more convenient for the purposes of addition or subtraction.

32. $\frac{3}{a-b}+\frac{2b+a}{b^2-a^2}$.

Changing the signs of the terms in the second denominator, at the same time changing the sign before the fraction (§ 116) (see Exercise 46), we have

$$\frac{3}{a-b}-\frac{2b+a}{a^2-b^2}.$$

The L. C. D. is now a^2-b^2.

Then,
$$\frac{3}{a-b}-\frac{2b+a}{a^2-b^2}=\frac{3(a+b)-(2b+a)}{a^2-b^2}$$
$$=\frac{3a+3b-2b-a}{a^2-b^2}=\frac{2a+b}{a^2-b^2}.$$

33. $\frac{1}{(x-y)(x-z)}-\frac{1}{(y-x)(y-z)}-\frac{1}{(z-x)(z-y)}$.

By § 117, we change the sign of the factor $y-x$ in the second denominator, at the same time changing the sign before the fraction; and we change the signs of both factors of the third denominator.

The expression then becomes

$$\frac{1}{(x-y)(x-z)}+\frac{1}{(x-y)(y-z)}-\frac{1}{(x-z)(y-z)}.$$

The L. C. D. is now $(x-y)(x-z)(y-z)$; then the result

$$=\frac{(y-z)+(x-z)-(x-y)}{(x-y)(x-z)(y-z)}=\frac{y-z+x-z-x+y}{(x-y)(x-z)(y-z)}.$$

$$=\frac{2y-2z}{(x-y)(x-z)(y-z)}=\frac{2(y-z)}{(x-y)(x-z)(y-z)}=\frac{2}{(x-y)(x-z)}.$$

34. $\dfrac{2}{3x-12}-\dfrac{3}{4-x}.$

35. $\dfrac{2a}{a^2-9}+\dfrac{5}{3-a}.$

36. $\dfrac{2e+3a}{2e-3a}+\dfrac{3e+4a}{4a-3e}.$

37. $\dfrac{x-1}{x^2-8x+15}+\dfrac{2-x}{x-5}.$

38. $\dfrac{v-b}{v-2a}-\dfrac{v+b}{v+2a}+\dfrac{b^2-4a^2}{4a^2-v^2}.$

39. $\dfrac{3x(a-b)}{x^2-b^2}-\dfrac{a-2b}{x+b}+\dfrac{a-b}{b-x}.$

40. $\dfrac{a}{k-a}-\dfrac{b}{k-b}-\dfrac{b^2-a^2}{b^2-bk}.$

41. $\dfrac{2(a+t)}{t}+\dfrac{a+t}{a-t}+\dfrac{t-a}{t+a}.$

42. $\dfrac{2u+7}{4-6u}+\dfrac{3u-5}{9u+6}-\dfrac{17u+2}{4-9u^2}.$

43. $\dfrac{m-2}{m-3}-\dfrac{3-m}{4-m}+\dfrac{m-5}{6-m}.$

MULTIPLICATION OF FRACTIONS

127. Required the product of $\frac{a}{b}$ and $\frac{c}{d}$.

Let $\qquad \frac{a}{b}\cdot\frac{c}{d}=x. \qquad (1)$

(Multiplication may be indicated by either $\times$ or $\cdot$.)

Multiplying both members by $b\cdot d$ (Ax. 7, § 4),

$$\frac{a}{b}\cdot\frac{c}{d}\cdot b\cdot d=x\cdot b\cdot d, \text{ or } \left(\frac{a}{b}\cdot b\right)\left(\frac{c}{d}\cdot d\right)=x\cdot b\cdot d;$$

for the factors of a product may be written in any order.

Now since the product of the quotient and the divisor gives the dividend (§ 60), we have

$$\frac{a}{b}\cdot b=a, \text{ and } \frac{c}{d}\cdot d=c.$$

Whence, $\qquad (a)(c)=x\cdot b\cdot d.$

Dividing both members by $b\cdot d$ (Ax. 8, § 4),

$$\frac{a\cdot c}{b\cdot d}=x. \qquad (2)$$

From (1) and (2), $\qquad \frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}.$

To multiply fractions, **multiply the numerators together for the numerator of the product, and the denominators for its denominator.**

128. Since c may be regarded as a fraction having the denominator 1, we have, by § 127,

$$\frac{a}{b}\cdot c=\frac{a}{b}\cdot\frac{c}{1}=\frac{ac}{b}.$$

Dividing both numerator and denominator by c (§ 115),

$$\frac{a}{b}\cdot c=\frac{a}{b\div c}.$$

Then, to multiply a fraction by a rational and integral expression, if possible, divide the denominator of the fraction by the expression; otherwise, multiply the numerator by the expression.

129. Common factors in the numerators and denominators should be cancelled before performing the multiplication.

Mixed expressions should be expressed in a fractional form (§ 122) before applying the rules.

1. Multiply $\dfrac{10\,a^3y}{9\,bx^2}$ by $\dfrac{3\,b^4x^3}{4\,a^3y^2}$.

$$\frac{10\,a^3y}{9\,bx^2}\cdot\frac{3\,b^4x^3}{4\,a^3y^2}=\frac{2\cdot 5\cdot 3\cdot a^3b^4x^3y}{3^2\cdot 2^2\cdot a^3bx^2y^2}=\frac{5\,b^3x}{6\,y}.$$

The factors cancelled are 2, 3, a^3, b, x^2, and y.

2. Multiply together $\dfrac{x^2+2\,x}{x^2+x-6}$, $2-\dfrac{x-4}{x-3}$, and $\dfrac{x^2-9}{x^2-4}$.

$$\frac{x^2+2\,x}{x^2+x-6}\cdot\left(2-\frac{x-4}{x-3}\right)\cdot\frac{x^2-9}{x^2-4}$$
$$=\frac{x^2+2\,x}{x^2+x-6}\cdot\frac{2\,x-6-x+4}{x-3}\cdot\frac{x^2-9}{x^2-4}$$
$$=\frac{x\cancel{(x+2)}}{\cancel{(x+3)}(x-2)}\cdot\frac{\cancel{x-2}}{\cancel{x-3}}\cdot\frac{\cancel{(x+3)}\cancel{(x-3)}}{\cancel{(x+2)}\cancel{(x-2)}}=\frac{x}{x-2}.$$

The factors cancelled are $x+2$, $x-2$, $x+3$, and $x-3$.

3. Multiply $\dfrac{a^2+b^2}{a^2-b^2}$ by $a-b$.

Dividing the denominator by $a-b$, $\dfrac{a^2+b^2}{a^2-b^2}\cdot(a-b)=\dfrac{a^2+b^2}{a+b}$.

4. Multiply $\dfrac{m}{m-n}$ by $m+n$.

Multiplying the numerator by $m+n$, $\dfrac{m}{m-n}\cdot(m+n)=\dfrac{m^2+mn}{m-n}$.

EXERCISE 52

Simplify the following:

1. $\dfrac{8\,am^2}{27\,b^4n^5}\times 9\,bn^5.$

2. $\dfrac{21\,a^3b^2}{8\,cd^6}\times\dfrac{4\,c^5d^6}{35\,a^3b^5}.$

3. $\dfrac{5\,a^2}{3\,b^2}\times\dfrac{9\,b^3}{10\,c^3}\times\dfrac{7\,c^4}{6\,a^4}.$

4. $\dfrac{3\,x^5}{10\,y^3}\cdot\dfrac{15\,y^7}{7\,z}\cdot\dfrac{28\,z^3}{9\,x^2}.$

5. $\dfrac{14\,b^3c}{15\,a^6}\cdot\dfrac{5\,c^3a}{12\,b^5}\cdot\dfrac{6\,a^2b}{7\,c^4}.$

6. $\dfrac{28\,m^7}{25\,n^5x^3}\cdot\dfrac{15\,n^6}{14\,m^5x^2}\cdot\dfrac{5\,x^8}{21\,m^3n^4}.$

7. $\dfrac{a^2-ab}{a^2+4\,a+4}\cdot(a+2).$

8. $\dfrac{5\,c+x}{x^2+4\,x-12}\cdot(x-2).$

9. $\dfrac{35\,a^2b}{4\,n^2-36\,d^2}\cdot\dfrac{n^2-6\,nd+9\,d^2}{20\,ab^2}.$

10. $\dfrac{a^2-2\,a-35}{2\,a^3-3\,a^2}\cdot\dfrac{4\,a^3-9\,a}{a-7}.$

11. $\dfrac{16\,z^2-9\,y^2}{8\,z^2+22\,zy-21\,y^2}\cdot\dfrac{2\,z^2+11\,zy+14\,y^2}{4\,z^2+11\,zy+6\,y^2}.$

12. $\dfrac{4\,t^2+4\,t+1}{3\,t+5}\cdot\dfrac{3\,tc+5\,c+6\,td+10\,d}{4\,t^2+10\,t+4}.$

13. $\dfrac{a^3-8\,b^3}{a^2-4\,b^2}\cdot\dfrac{a^2+4\,ab+4\,b^2}{2\,a^3+4\,a^2b+8\,ab^2}.$

14. $\left(4-\dfrac{a^2+2\,ab+b^2}{ab}\right)\left(\dfrac{4\,ab}{a^2-2\,ab+b^2}+1\right).$

15. $\dfrac{3\,x-4\,y}{3\,x+4\,y}\cdot\dfrac{9\,x^2+24\,xy+16\,y^2}{3\,x^2-4\,xy}+2.$

Note: In problems similar to example 15, indicated multiplication or division must be performed before addition or subtraction is made. 2 is to be added to the product of the fractions, not to the second fraction.

For example, in $13+4\times3+6\div2-4$, 4×3 and $6\div2$ must be performed before uniting the terms of the expression.

16. $\dfrac{25\,m^2-40\,m+16}{4\,m^2-9}\cdot\dfrac{8\,m^3-12\,m^2}{25\,m^2-16}-2\,m+\dfrac{8}{2\,m+3}.$

DIVISION OF FRACTIONS

130. Required the quotient of $\frac{a}{b}$ divided by $\frac{c}{d}$.

Let $$\frac{a}{b}\div\frac{c}{d}=x. \quad (1)$$

Then since the dividend is the product of the divisor and quotient (§ 60), we have
$$\frac{a}{b}=\frac{c}{d}\times x.$$

Multiplying both members by $\frac{d}{c}$ (Ax. 7, § 4),
$$\frac{a}{b}\times\frac{d}{c}=\frac{c}{d}\times x\times\frac{d}{c}=x. \quad (2)$$

From (1) and (2), $$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}. \quad \text{(Ax. 4, § 4)}$$

Then, to divide one fraction by another, multiply the dividend by the divisor inverted.

If the divisor is an integer, c may be regarded as a fraction having the denominator 1.

Mixed expressions should be expressed in a fractional form (§ 122) before applying the rules.

1. Divide $\frac{6\,a^2b}{5\,x^3y^4}$ by $\frac{9\,a^2b^3}{10\,x^2y^7}$.

We have, $$\frac{6\,a^2b}{5\,x^3y^4}\div\frac{9\,a^2b^3}{10\,x^2y^7}=\frac{6\,a^2b}{5\,x^3y^4}\cdot\frac{10\,x^2y^7}{9\,a^2b^3}=\frac{4\,y^3}{3\,b^2x}.$$

2. Divide $2-\frac{2\,x-3}{x+1}$ by $3-\frac{3\,x^2-13}{x^2-1}$.

$$\left(2-\frac{2\,x-3}{x+1}\right)\div\left(3-\frac{3\,x^2-13}{x^2-1}\right)$$
$$=\frac{2\,x+2-2\,x+3}{x+1}\div\frac{3\,x^2-3-3\,x^2+13}{x^2-1}$$
$$=\frac{5}{x+1}\cdot\frac{x^2-1}{10}=\frac{5(x+1)(x-1)}{2\cdot 5\cdot(x+1)}=\frac{x-1}{2}.$$

3. Divide $\frac{m^3-n^3}{m^2+n^2}$ by $m-n$.

Dividing the numerator by $m-n$, $\frac{m^3-n^3}{m^2+n^2}\div(m-n)=\frac{m^2+mn+n^2}{m^2+n^2}$.

4. Divide $\frac{a^2+b^2}{a-b}$ by $a+b$.

Multiplying the denominator by $a+b$, $\frac{a^2+b^2}{a-b} \div (a+b) = \frac{a^2+b^2}{a^2-b^2}$.

If the numerator and denominator of the divisor are exactly contained in the numerator and denominator, respectively, of the dividend, it follows from § 127 that the numerator of the quotient may be obtained by dividing the numerator of the dividend by the numerator of the divisor; and the denominator of the quotient by dividing the denominator of the dividend by the denominator of the divisor.

5. Divide $\frac{9x^2-4y^2}{x^2-y^2}$ by $\frac{3x+2y}{x-y}$.

We have, $\frac{9x^2-4y^2}{x^2-y^2} \div \frac{3x+2y}{x-y} = \frac{3x-2y}{x+y}$.

EXERCISE 53

Simplify the following:

1. $\frac{45x^6m^2}{4y^5n} \div 9x^3m^4$.

2. $\frac{12a^6b^2}{55c^3d^5} \div \frac{9a^4b^8}{22c^7d^5}$.

3. $\frac{9x^2-16}{3x+7} \div (3x-4)$.

4. $\frac{4m-k}{2m+3k} \div (4m+k)$.

5. $\frac{t^2-t-12}{8t} \div \frac{t^2-8t+16}{6t}$.

6. $\frac{2v^2-5v-12}{3v+2} \div \frac{4v^2-10v-24}{9v^2-4}$.

7. Divide $\frac{2ac-2bc+3ad-3bd}{4c^2+4cd+d^2}$ by $\frac{a^2+5ab-6b^2}{4c^2-4cd-3d^2}$.

8. $\frac{a^3+27}{a^4+4} \div \frac{a^2b-3ab+9b}{a^2+2a+2}$.

9. $\frac{5t^2+8tu+3u^2}{t^3+u^3} \div \frac{3t^2+7tu+4u^2}{t+u}$.

10. $2-\frac{3x-3}{4x+7} \div 3-\frac{4x-26}{3x-3}$.

11. Divide $2-\frac{3x-3}{4x+7}$ by $3-\frac{4x-26}{3x-3}$.

12. $\left(\frac{4t-3}{t+2}-5t+2\right) \div \left(\frac{4t^2-2t-9}{t+2}+t+4\right)$.

COMPLEX FRACTIONS

131. **A Complex Fraction** is a fraction having one or more fractions in either or both of its terms.

It is simply a case in division of fractions; its numerator being the dividend, and its denominator the divisor.

1. Simplify $\dfrac{a}{b-\dfrac{c}{d}}$.

$$\frac{a}{b-\dfrac{c}{d}}=\frac{a}{\dfrac{bd-c}{d}}=a\times\frac{d}{bd-c}\ (\S\,130)=\frac{ad}{bd-c}.$$

It is often advantageous to simplify a complex fraction by multiplying its numerator and denominator by the L. C. M. of their denominators (§ 115).

2. Simplify $\dfrac{\dfrac{a}{a-b}-\dfrac{a}{a+b}}{\dfrac{b}{a-b}+\dfrac{a}{a+b}}$.

The L. C. M. of $a+b$ and $a-b$ is $(a+b)(a-b)$.
Multiplying both terms by $(a+b)(a-b)$, we have

$$\frac{\dfrac{a}{a-b}-\dfrac{a}{a+b}}{\dfrac{b}{a-b}+\dfrac{a}{a+b}}=\frac{a(a+b)-a(a-b)}{b(a+b)+a(a-b)}=\frac{a^2+ab-a^2+ab}{ab+b^2+a^2-ab}=\frac{2\,ab}{a^2+b^2}.$$

EXERCISE 54

Simplify the following:

1. $\dfrac{\dfrac{1}{r}+\dfrac{1}{s}}{\dfrac{1}{r}-\dfrac{1}{s}}$.

2. $\dfrac{\dfrac{r^2+u^2}{ru}}{\dfrac{1}{r}+\dfrac{1}{u}}$.

3. $\dfrac{1-\dfrac{3\,a}{7\,b}}{b-\dfrac{9\,a^2}{49\,b}}$.

4. $\dfrac{c+\dfrac{8\,d^3}{c^2}}{1+\dfrac{2\,d}{c}}$.

5. $\dfrac{\dfrac{x-y}{x}-\dfrac{x+y}{y}}{\dfrac{x+y}{x}+\dfrac{x-y}{y}}$.

6. $\dfrac{\dfrac{2\,a}{3\,c}+2+\dfrac{3\,c}{2\,a}}{\dfrac{1}{3\,c}+\dfrac{1}{2\,a}}$.

7. $\dfrac{\dfrac{a^2-2a-35}{2a^3-3a^2}}{\dfrac{a-7}{4a^3-9a}}$.

8. $\dfrac{\dfrac{4m^2+8m+3}{2m^2-5m+3}}{\dfrac{4m^2-1}{6m^2-9m}}$.

9. $\dfrac{\dfrac{1}{x+2}+\dfrac{2}{x+3}}{\dfrac{x^2-3x-10}{x+2}}$.

10. $\dfrac{\dfrac{x-2}{x+3}-\dfrac{x+2}{x-3}}{\dfrac{x}{x+3}-\dfrac{x}{x-3}}$.

11. $\dfrac{\dfrac{a^2-b^2-c^2}{2bc}-1}{\dfrac{a^2+b^2-c^2}{2ab}+1}$.

12. $\dfrac{\dfrac{e^4+e^2f^2+f^4}{e^3+f^3}}{\dfrac{e^2+ef+f^2}{e^2-f^2}}$.

13. $\dfrac{\dfrac{4v^2-23v+6}{v^2-7v+12}-2}{3-\dfrac{5v^2-72}{v^2+3v-18}}$.

14. $\dfrac{\left(x-\dfrac{3}{x}\right)^2+16}{\left(x+\dfrac{3}{x}\right)^2+4}$.

15. $\dfrac{\dfrac{x^2-y^2-2yz-z^2}{9t^2-25u^2}}{\dfrac{(x+y+z)(x-y-z)}{6t+10u}}$.

16. $\dfrac{\dfrac{5h^2+14hk-3k^2}{8h^2-6hk-35k^2}}{\dfrac{5h^2-11hk+2k^2}{4h^2-hk-14k^2}}$.

MISCELLANEOUS AND REVIEW EXAMPLES

EXERCISE 55

Simplify the following:

1. $\dfrac{3a-2b}{a^2-4b^2}\cdot(a-2b)$.

2. $\dfrac{8c^3+27d^3}{8c^3-27d^3}\cdot(2c-3d)$.

3. $\left(6a^2+\dfrac{b}{5}\right)\div\left(c^2-\dfrac{x-a}{2}\right)$.

4. $x+\dfrac{x+1}{a}\cdot\dfrac{x-1}{a+b}$.

5. $\left(x+\dfrac{x+1}{a}\right)\left(\dfrac{x-1}{a+b}\right)$.

6. $\dfrac{x^4-b^4}{x^2-2bx+b^2}\div\dfrac{x^2+bx}{x-b}$.

7. From $3a+\dfrac{a}{b}$ take $a-\dfrac{a-c}{d}$.

8. Find the sum of $c-\frac{6b}{8a}$ and $1-\frac{c}{a}$.

9. Find the sum of $3a+\frac{2ax}{x-a}$ and $2a-\frac{-x}{a+b}$.

Simplify:

10. $a-b-\frac{a^2+b^2}{a-b}$.

11. $a+b-\frac{(a-b)^2}{a+b}$.

12. $a+b+\frac{(a-b)^2}{a+b}$.

13. $\frac{x^4-3x^3-8x+24}{x^4+3x^3-8x-24}$.

14. $\frac{(a+2c)^2-(b-3)^2}{(2c+3)^2-(a-b)^2}$.

15. $\frac{\frac{1}{3x}+\frac{1}{2y}}{9x^2-4y^2}+\frac{\frac{1}{2x}-\frac{1}{3y}}{4x^2-9y^2}$.

16. $\frac{x(x^2+x+1)-2(x^2+x+1)}{x^4+x^2+1}$.

17. $\left(\frac{a+2}{a}+\frac{2}{a-3}\right)\left(\frac{a}{a-2}-\frac{3}{a+3}\right)$.

18. $\frac{1+e^2+\frac{e^4}{1-e^2}}{1-e^2}$.

19. $\frac{1+\frac{c^2}{x^2-a^2}}{x^2-a^2+c^2}$.

20. $\frac{\frac{a}{x^2}}{1+\frac{a^2}{x^2}}+\frac{\frac{a}{(x-a)^2}}{\frac{x+a}{x-a}}$.

21. $\frac{1}{x-1}-\frac{1}{x-2}+\frac{1}{x-3}-\frac{1}{x-4}$.

(Combine the first two fractions, then the last two, and add the results.)

22. $\frac{3}{2n+1}+\frac{3}{2n-1}-\frac{5n^2}{8n^3+1}-\frac{5n^2}{8n^3-1}$.

23. $\frac{a}{a+3}+\frac{1}{a-3}-\frac{3}{a^2-9}-\frac{a^2+2a}{a^2+9}$.

(First add the first two fractions, to the result add the third fraction and to this result add the last fraction.)

24. $\dfrac{3a}{a+b}+\dfrac{3a}{a-b}+\dfrac{6a^2}{a^2+b^2}+\dfrac{12a^4}{a^4+b^4}$.

25. $\dfrac{1}{x^2+2x-3}-\dfrac{1}{x^2+x-6}+\dfrac{1}{x^2-3x+2}$.

26. $\dfrac{x-2}{2x^2-13x-45}-\dfrac{x+3}{2x^2+29x+60}+\dfrac{4x-1}{x^2+3x-108}$.

27. Translate into English:

Is this Spanish?

$$\frac{4a-b}{2a+b}-5c\times 2a-\frac{3a+b}{c}.$$

28. Translate into English:

$$\left(\frac{4a-b}{2a+b}-5c\right)\left(2a-\frac{3a+b}{c}\right).$$

29. Translate into English:

$$\frac{3m-1}{2a-1}-4\div 2-\frac{4m-1}{3a-1}.$$

30. Translate into English:

$$\left(\frac{3m-1}{2a-1}-4\right)\div\left(2-\frac{4m-1}{3a-1}\right).$$

31. State algebraically: The sum of $3a$ and b divided by the difference between $3a$ and b: Multiply this quotient by the fraction whose numerator is the difference between $9a^2$ and b^2 and whose denominator is the sum of $9a^2$ and b^2. Reduce your statement.

X. FRACTIONAL EQUATIONS. RATIO AND PROPORTION

SOLUTION OF FRACTIONAL EQUATIONS

132. If a fraction whose numerator is a polynomial is preceded by a $-$ sign, it is convenient, on clearing of fractions, to write the numerator in parenthesis, as shown in Ex. 1.

If this is not done, care must be taken to *change the sign of each term of the numerator* when the denominator is removed. This is readily understood if one remembers that the line between the numerator and denominator acts as a *vinculum.*

1. Solve the equation $\frac{3t-1}{4}-\frac{4t-5}{5}=4+\frac{7t+5}{10}$.

The L. C. M. of 4, 5, and 10 is 20.
Multiplying each term by 20, we have

$$15t-5-(16t-20)=80+14t+10.$$

Whence, $15t-5-16t+20=80+14t+10.$
Transposing, $15t-16t-14t=80+10+5-20.$
Uniting terms, $-15t=75.$
Dividing by -15, $t=-5.$
Verify the result.

2. Solve the equation $\frac{2}{x-2}-\frac{5}{x+2}-\frac{2}{x^2-4}=0$.

The L. C. M. of $x-2$, $x+2$, and x^2-4 is x^2-4.
Multiplying each term by x^2-4, we have

$$2(x+2)-5(x-2)-2=0.$$

Or, $2x+4-5x+10-2=0.$
Transposing, and uniting terms, $-3x=-12$, and $x=4$.
Verify the result.

If the denominators are partly monomial and partly polynomial, it is often advantageous to clear of fractions at first partially; multiplying each term of the equation by the L. C. M. of the *monomial* denominators.

3. Solve the equation $\frac{6s+1}{15}-\frac{2s-4}{7s-16}=\frac{2s-1}{5}$.

Multiplying each term by 15, the L. C. M. of 15 and 5,

$$6s+1-\frac{30s-60}{7s-16}=6s-3.$$

Transposing, and uniting terms, $4=\frac{30s-60}{7s-16}.$
Clearing of fractions, $28s-64=30s-60.$
Then, $-2s=4$, and $s=-2.$
Verify the result.

EXERCISE 56

Solve the following equations, verifying each result:

1. $\frac{1}{3}+\frac{2}{5x}=\frac{13}{15}-\frac{2}{3x}.$

2. $\frac{1}{x}+\frac{2}{3x}-\frac{3}{5x}-\frac{2}{15x}=\frac{7}{30}.$

3. $\frac{4}{x}-\frac{1}{4x}-\frac{6}{7x}-\frac{2}{3x}=\frac{187}{168}.$

4. $3x-\frac{5x+15}{6}=\frac{4x}{3}.$

5. $\frac{5}{3}t-\frac{4t-11}{5}=3t+\frac{1}{15}.$

6. $\frac{4v}{5}+\frac{2v-7}{4}+\frac{1}{12}=\frac{7v}{15}.$

7. $R-\frac{4R+7}{3}+\frac{5R+9}{8}=-\frac{3}{2}.$

8. $\frac{3m-1}{4m}-\frac{5m+1}{8m}+\frac{9}{40}-\frac{8m-5}{5m}=0.$

9. $\frac{5}{u}-\frac{1}{21}-\frac{5(11-3u)}{6u}=\frac{7-9u}{2u}-\frac{5}{7u}.$

10. $\frac{5(x-1)}{6}-\frac{2(x+2)}{3}=4-\frac{5x-15}{4}.$

11. $\frac{6g+4}{8}-\frac{3g-4}{2}+1=\frac{5g+8}{9}.$

12. $\frac{2v+1}{7}-\frac{6v-4}{9v+1}=\frac{4v-5}{14}.$

13. $\frac{8x+15}{3}-\frac{11x+15}{5}-\frac{13x+29}{10}=\frac{14x+66}{15}.$

14. $\frac{2t+9}{4t+1}=\frac{t-11}{2t-15}.$

15. $\frac{3x-1}{18x-19}=\frac{x+1}{6x-7}.$

16. $\frac{24}{x-2}-\frac{24}{x}=1.$ (See § 103.)

17. $\frac{t-2}{t+5}-\frac{t+4}{t-3}=-\frac{7}{3}.$

18. $\frac{5}{5-R}+\frac{8}{8-R}=3.$

19. $\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-3}{x-4}-\frac{x-4}{x-5}$

(Reduce each fraction to a mixed number, then see example 21, Exercise 55.)

20. $\dfrac{2x}{x+1}+\dfrac{2x+1}{x}=\dfrac{3x+2}{x^2+x}$.

21. What number added to twice its reciprocal gives 3?

22. The sum of $\frac{1}{3}$ of a certain number and $\frac{1}{3}$ of its square is $\frac{1}{4}$. Find the number.

23. Make two problems similar to 21 and 22.

133. Solution of Special Forms of Fractional Equations.

1. Solve the equation $\dfrac{2x-1}{2x-3}+\dfrac{x^2-x}{x^2+4}=2$.

We divide each numerator by its corresponding denominator; then

$$1+\frac{2}{2x-3}+1-\frac{x+4}{x^2+4}=2, \text{ or } \frac{2}{2x-3}-\frac{x+4}{x^2+4}=0.$$

Clearing of fractions, $2x^2+8-(2x^2+5x-12)=0$.

Then, $2x^2+8-2x^2-5x+12=0$; whence, $x=4$.

We reject a root which does not satisfy the given equation.

2. Solve the equation $\dfrac{1}{x-3}+\dfrac{1}{x-2}=\dfrac{3x-7}{x^2-5x+6}$.

Multiplying both members by $(x-3)(x-2)$, or x^2-5x+6,

$$x-2+x-3=3x-7.$$

Transposing, and uniting terms, $-x=-2$, or $x=2$.

If we substitute 2 for x, the fraction $\dfrac{1}{x-2}$ becomes $\dfrac{1}{0}$.

Since division by 0 is impossible, the solution $x=2$ does not satisfy the given equation, and we reject it; the equation has no solution.

3. Solve the equation $\dfrac{3}{x+10}+\dfrac{4}{x+6}=\dfrac{2}{x+8}+\dfrac{5}{x+9}$.

Adding the fractions in each member, we have

$$\frac{7x+58}{(x+10)(x+6)}=\frac{7x+58}{(x+8)(x+9)}.$$

Clearing of fractions, and transposing all terms to the first member,

$$(7x+58)(x+8)(x+9)-(7x+58)(x+10)(x+6)=0. \quad (1)$$

Factoring, $(7x+58)[(x+8)(x+9)-(x+10)(x+6)]=0$.

Expanding, $(7x+58)(x^2+17x+72-x^2-16x-60)=0$.

Or, $(7x+58)(x+12)=0$.

This equation may be solved by the method of § 103.

Placing $7x+58=0$, we have $x=-\dfrac{58}{7}$.

Placing $x+12=0$, we have $x=-12$.

134. If we should solve equation (1), in Ex. 3 of § 133, by dividing both members by $7x+58$, we should have

$$(x+8)(x+9)-(x+10)(x+6)=0.$$

Then, $\quad x^2+17x+72-x^2-16x-60=0$, or $x=-12$.

In this way, the solution $x=-\frac{58}{7}$ *is lost.*

It follows from this that it is never allowable to divide both members of an equation by any expression which involves the unknown numbers, unless the expression be retained as a factor as in § 103 and the root preserved, for in this way solutions are lost.

EXERCISE 57

Solve the following equations:

1. $\dfrac{4x+11}{x^2+x-20}=\dfrac{1}{x+5}-\dfrac{1}{x-4}.$

2. $\dfrac{x+3}{x+2}+\dfrac{x+4}{x+3}+\dfrac{x+2}{x+4}=3.$

3. $\dfrac{3}{x+9}+\dfrac{2}{x+4}=\dfrac{1}{x+3}+\dfrac{4}{x+18}.$

4. $\dfrac{2x+3}{2x-3}-\dfrac{2x-3}{2x+3}-\dfrac{36}{4x^2-9}=0.$

5. $\dfrac{2x+5}{x+7}-\dfrac{3x^2+24x+19}{x^2+8x+7}=-1.$

6. $\dfrac{x^2-2x+5}{x^2-2x-3}+\dfrac{x^2+3x-7}{x^2+3x+1}=2.$

SOLUTION OF LITERAL LINEAR EQUATIONS

135. A Literal Equation is one in which some or all of the known numbers are represented by letters; as,

$$2x+a=b^2+10.$$

Ex. Solve the equation $\dfrac{x}{x-a}-\dfrac{x+2b}{x+a}=\dfrac{a^2+b^2}{x^2-a^2}.$

Multiplying each term by x^2-a^2,

$$x(x+a)-(x+2b)(x-a)=a^2+b^2,$$

or, $$x^2+ax-(x^2+2bx-ax-2ab)=a^2+b^2,$$

or, $$x^2+ax-x^2-2bx+ax+2ab=a^2+b^2,$$

or, $$2ax-2bx=a^2-2ab+b^2.$$

Factoring both members, $$2x(a-b)=(a-b)^2.$$

Dividing by $2(a-b)$, $$x=\frac{(a-b)^2}{2(a-b)}=\frac{a-b}{2}.$$

In solving fractional literal equations, we must reject any solution which does not satisfy the given equation. Compare Ex. 2, § 133.

EXERCISE 58

1. Find the coefficient of x in
$$(b+x)^2+(2c-3ax)^2.$$

2. Find the coefficient of t^2 in
$$a(t-b)(t-b)-b(t-a)(t-a)-3at(2a+t).$$

3. $\dfrac{5x-a}{2x+a}+\dfrac{2x+5a}{4x}=3.$ Solve for x.

4. $\dfrac{v}{ab}+\dfrac{v}{bc}+\dfrac{v}{ca}=a+b+c.$ Solve for v.

5. $\dfrac{c^2t+b}{ct}-\dfrac{b^2t+c}{bt}=\dfrac{c^3b-cb^3+b^2-c^2}{cbt}.$ Solve for t.

6. $\dfrac{2m+3b}{2m-3b}-\dfrac{4a+5b}{4a-5b}=0.$ Solve for m.

7. $\dfrac{u(a+4b)-b^2}{a^2-b^2}+\dfrac{u-b}{a+b}=\dfrac{u+a}{a-b}.$ Solve for u.

8. $\dfrac{a-b}{t-c}+\dfrac{b-c}{t-a}+\dfrac{c-a}{t}=0.$ Solve for t.

9. $\dfrac{5}{2s+5d}-\dfrac{2}{3s-4d}=\dfrac{3d}{6s^2+7ds-20d^2}.$ Solve for s.

10. $\dfrac{a}{w-a}-\dfrac{b}{w-b}=\dfrac{b^2-a^2}{b^2-bw}.$ Solve for w.

11. $\dfrac{a(x-a)}{x-b}+\dfrac{b(x-b)}{x-a}=a+b.$ Solve for x.

12. $\dfrac{4v+3n}{v+2n}+\dfrac{4v-3n}{3n-v}=-\dfrac{10n^2}{v^2-nv-6n^2}$. Solve for n.

13. $\dfrac{3x}{2}-\dfrac{5ax-2b}{4a}=\dfrac{a+3bx}{8b}-\dfrac{ax+2a^2-4b}{16ab}$.

14. $\dfrac{2R+3a}{2R-3a}=\dfrac{3a+4b}{3a-4b}$. Solve for R.

15. $\dfrac{ax-b}{bx}+\dfrac{bx+a}{ax}=2+\dfrac{a-b}{abx}$. Solve for x.

RATIO AND PROPORTION

RATIO

136. *The* **Ratio** *of one whole or fractional number*, a, *to another*, b, *is the quotient of* a *divided by* b. Thus, the ratio of a to b is $\frac{a}{b}$; it is also expressed $a:b$. We make no attempt to define *ratio*. When applied to whole or fractional numbers, ratio is only another name for quotient or fraction. When the fraction $\frac{a}{b}$ is called a ratio, its numerator a is called the *antecedent* or *first term*, and its denominator b is called the *consequent* or *second term*.

The ratios here spoken of are but fractions under another name, and *have all the properties of fractions*.

If a and b are positive numbers, and $a>b$, $\frac{a}{b}$ is called a *ratio of greater inequality*; if $a<b$, it is called a *ratio of less inequality*.

(The signs $>$ and $<$ are read "is greater than" and "is less than" respectively.)

PROPORTION

137. A **Proportion** is an equation whose members are equal ratios.

Thus, if $\frac{a}{b}$ and $\frac{c}{d}$ are equal ratios,

$$a:b=c:d, \text{ or } \frac{a}{b}=\frac{c}{d},$$

is a proportion. (See example 14, Exercise 58.)

138. In the above proportion, a is called the *first term*, b the *second*, c the *third*, and d the *fourth*.

The first and third terms of a proportion are called the *antecedents*, and the second and fourth terms the *consequents*.

The first and fourth terms are called the *extremes*, and the second and third terms the *means*.

139. If the means of a proportion are equal, either mean is called the **Mean Proportional** between the first and last terms, and the last term is called the **Third Proportional** to the first and second terms.

Thus, in the proportion $\frac{a}{b}=\frac{b}{c}$, b is the mean proportional between a and c, and c is the third proportional to a and b.

The **Fourth Proportional** to three numbers is the fourth term of a proportion whose first three terms are the three numbers taken in their order.

Thus, in the proportion $\frac{a}{b}=\frac{c}{d}$, d is the fourth proportional to a, b, and c.

140. A **Continued Proportion** is a series of equal ratios, in which each consequent is the same as the next antecedent; as,

$$a:b=b:c=c:d=d:e$$

or

$$\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e}.$$

The definitions and explanations in §§ 138 and 139 refer to proportions written in the form

$$a:b=c:d.$$

Because of greater facility in operation, however, we shall use the form

$$\frac{a}{b}=\frac{c}{d}.$$

IMPORTANT PROPERTIES OF PROPORTIONS

141. In any proportion, the product of the extremes is equal to the product of the means.

Let the proportion be $\frac{a}{b}=\frac{c}{d}$.

Clearing of fractions, $ad=bc$.

142. (Converse of § 141.) **If the product of two numbers be equal to the product of two others, one pair may be made the extremes, and the other pair the means, of a proportion.**

Let $$ad=bc.$$

Dividing by bd, $$\frac{ad}{bd}=\frac{bc}{bd}, \text{ or } \frac{a}{b}=\frac{c}{d}.$$

143. In any proportion, the terms are in proportion by Alternation; that is, the means can be interchanged.

In § 142, had we divided by cd, the proportion would have been

$$\frac{a}{c}=\frac{b}{d}.$$

In like manner, the extremes can be interchanged.

144. In any proportion, the terms are in proportion by Inversion; that is, the second term is to the first as the fourth term is to the third.

It follows from § 144 that, in any proportion, the means can be written as the extremes, and the extremes as the means.

145. In any proportion, the terms are in proportion by Composition; that is, the sum of the first two terms is to the first term as the sum of the last two terms is to the third term.

Let the proportion be $\frac{a}{b}=\frac{c}{d}$. Then $\frac{a+b}{a}=\frac{c+d}{c}$,

also, $$\frac{a+b}{b}=\frac{c+d}{d}.$$

146. In any proportion, the terms are in proportion by Division; that is, the difference between the first two terms is to the first term as the difference between the last two terms is to the third term.

Let the proportion be $\frac{a}{b}=\frac{c}{d}$. Then $\frac{a-b}{a}=\frac{c-d}{c}$,

also, $$\frac{a-b}{b}=\frac{c-d}{d}.$$

147. In any proportion, the terms are in proportion by Composition and Division; that is, the sum of the first two terms is to their difference as the sum of the last two terms is to their difference.

Let the proportion be $\frac{a}{b}=\frac{c}{d}$. Then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

148. In any number of proportions, the products of the corresponding terms are in proportion.

Let the proportions be $\frac{a}{b}=\frac{c}{d}$, and $\frac{e}{f}=\frac{g}{h}$.

Then, $\frac{ae}{bf}=\frac{cg}{dh}$.

149. In a series of equal ratios, any antecedent is to its consequent as the sum of all the antecedents is to the sum of all the consequents.

Let, $a:b=c:d=e:f$.

Then, $\frac{a}{b}=\frac{a+c+e}{b+d+f}$.

EXERCISE 59

The following problems lead both to integral and fractional equations. *Always verify results.* In verifying results obtained from written problems it is sufficient to ascertain if results satisfy the conditions stated in the problem.

1. m is a mean proportional between 2 and 8; find m.

2. x is a positive integer. If x be added to both terms of the ratio $\frac{3}{5}$, what is the effect on the ratio? If $\frac{5}{3}$ were the ratio would the effect be the same?

3. In any proportion if the first antecedent and its consequent be multiplied by m, is the proportion changed? Why?

4. d is a fourth proportional to 3, 4, and 12; find d.

5. A can do in 8 days a piece of work which B can perform in 10 days. In how many days can it be done by both working together?

Let $x=$ the number of days required.

Then, $\frac{1}{x}=$ the part both can do in one day.

Also, $\frac{1}{8}=$ the part A can do in one day,

and $\frac{1}{10}=$ the part B can do in one day.

By the conditions, $\frac{1}{8}+\frac{1}{10}=\frac{1}{x}$.

Clearing of fractions, $5x+4x=40$, or $9x=40$.

Whence, $x=4\frac{4}{9}$, the number of days required.

6. A piece of work can be done by A in 7 hours, and by B in 6 hours; in how many hours can the work be done by both working together?

7. The numerator of a certain fraction is 4 greater than the denominator. If 5 be added to both numerator and denominator, the result is $\frac{3}{2}$. Find the fraction.

(Hint: Let $d =$ the denominator.)

8. Given two numbers, 5 and 2. What number must you add to each so that the first sum may be $\frac{4}{5}$ of the second sum?

9. A's age is 4 years more than $\frac{3}{5}$ of B's, and the sum of their ages is 44. Find the age of each.

Write in the form of proportion:

10. $x^2+3x+2=a^2-a-12.$

11. $x^2-6x+9=a^2-9.$

12. $4c^2+12c+9=d^2-4d+4.$

13. $x(x+3)=y(y-7).$

14. $x(y-3)=y(x-3).$

15. Solve, using composition and division:

$$\frac{2t-1}{2t+1}=\frac{a-b}{a+b}.$$

16. Use composition and division, then solve:

$$\frac{2x+3}{2x-4}=\frac{x+2}{x-2}.$$

17. Write this proportion as a simple equation:

$$\frac{x-4}{y+2}=\frac{a+2}{a-4}.$$

18. The first digit of a number exceeds the second by 3; and if the number, increased by 4, be divided by the sum of its digits, the quotient is 8. Find the number.

(Let $x =$ the second digit, the number itself is ten times the first digit, plus the second digit.)

19. A piece of work can be done by A and B working together in 10 days. After working together 7 days, A leaves,

and B finishes the work in 9 days. How long would A alone have taken to do the work?

20. Solve this proportion: $\frac{x-1}{x+1}=\frac{2}{3}$.

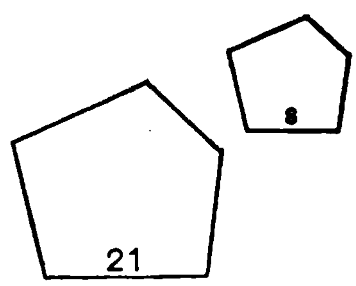
21. Simplify, using composition and division:

$$\frac{a-1}{a+1}=\frac{c-1}{c+1}.$$

22. $\frac{t+(a+b)}{t-(a+b)}=\frac{a}{b}$. Solve for t.

23. The volumes of two spheres are proportional to the cubes of their like dimensions. If a sphere 6 inches in diameter weighs 351 ounces, what is the weight of a sphere of the same material whose diameter is 10 inches?

24. In similar figures in geometry, like lines are proportional. The sums of the sides of two similar polygons are 119 and 68 respectively. If a side of the first polygon is 21, what is the corresponding side of the second?



25. A garrison of 700 men has provisions for 11 days. After 3 days a certain number of men leave, and the provisions last 10 days after this time. How many men leave?

26. The digits of a certain number are three consecutive numbers, of which the middle digit is the least, and the last digit is the greatest. If the number be divided by the sum of the digits the quotient is 36. What is the number?

27. If b is a mean proportional between a and c, show that $\frac{a}{c}=\frac{a^2}{b^2}$.

28. The angles of a triangle ABC are in proportion to 1, 2, 3. The sum of the angles of a triangle is 180°. How many degrees in each angle? See note, Ex. 31.

29. How many degrees in each angle of the triangle ABC, if angle B is twice angle A, and angle C is 20° more than B?

30. B can do a piece of work in $\frac{3}{4}$ as many days as A, and C can do it in $\frac{8}{9}$ as many days as B; together they can do the work in $3\frac{3}{23}$ days. In how many days can each alone do the work?

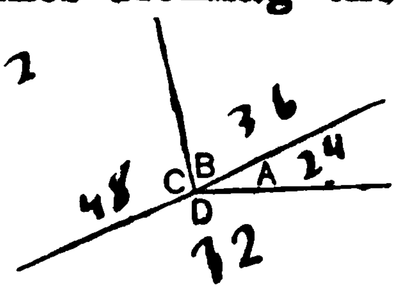
31. Two persons, A and B, 63 miles apart, start at the same time and travel toward each other. A travels at the rate of 4 miles an hour, and B at the rate of 3 miles an hour. How far will each have travelled when they meet?

It is often advantageous to represent the unknown number by some *multiple* of a letter.

Then let $4x$ = the number of miles that A travels,
and $3x$ = the number of miles that B travels.

32. A man started from his home to catch a train at the rate of one yard in a second, and arrived 2 minutes late. If he had walked at the rate of 4 yards in 3 seconds, he would have been $3\frac{1}{2}$ minutes too early. Find the distance to the station.

33. From a point O are drawn four lines forming the angles A, B, C, and D. The sum of these angles is 360°, and they are in the proportion of 2, 3, 4, and 6. Find each angle.



34. Find the mean proportional between

$$\frac{x^2-x-12}{x-5} \text{ and } \frac{x^2-9x+20}{x+3}.$$

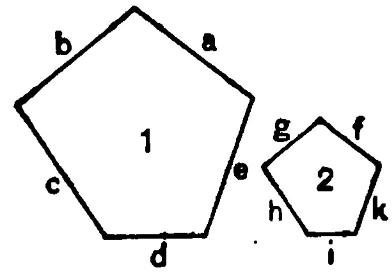
35. Use § 147, then solve for m:

$$\frac{3m-8}{3m+5}=\frac{2m-6}{2m+7}.$$

36. The numerator of a fraction exceeds the denominator by 5. If the numerator be decreased by 9, and the denominator increased by 6, the sum of the resulting fraction and the given fraction is 2. Find the fraction.

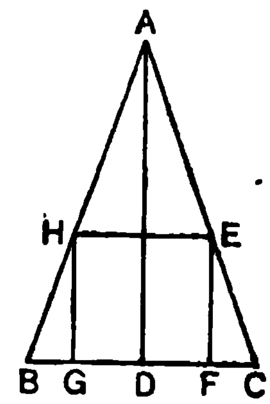
37. The areas of two similar pentagons (figures having five sides) are proportional to the squares of any two cor-

responding sides. The area of the first pentagon is 240, and the side a is 10; the area of the second pentagon is 60. Find the side f.



38. The width of a field is $\frac{2}{3}$ its length. If the width were increased by 5 feet, and the length by 10 feet, the area would be increased by 400 square feet. Find the dimensions.

39. A rectangle, $GHEF$, whose base, GF, is 6, is inscribed in an isosceles triangle, ABC, whose altitude, AD, is 14, and whose base, BC, is 10. Knowing from geometry that $\frac{AD}{DC}=\frac{EF}{FC}$, $BD=DC$, and that $GD=DF$, find EF.



40. $.2\,x+.001-.03\,x=.113\,x-.0161.$

Transposing, $.2\,x-.03\,x-.113\,x=-.0161-.001.$

Uniting terms, $.057\,x=-.0171.$

Dividing by .057, $x=-.3.$

41. $7.98\,x-3.75=.23\,x+.125.$

42. $3\,t+.052-7.8\,t=.04-5.82\,t-.0696.$

43. $.05\,v-1.82-.7\,v=.008\,v-.504.$

44. $.73\,D+8.86=.6(2.3\,D-.4).$

45. $.07(8\,s-5.7)=.8(5\,s+.86)+1.321.$

*46. The density of a substance is defined as the number of grams in one cubic centimeter. Hence the total number of grams, M, in any body is equal to its density, D, multiplied by its volume, V; or, to state this relation algebraically,

$$M=DV,$$

V being given in cubic centimeters, and D in grams.

Two blocks, one of iron and one of copper, weigh the same number of grams; the iron has a volume of 10 cubic centi-

* Note: The metric tables will be found on page 228.

meters and a density of 7.4; the copper has a density of 8.9. Find the volume of the copper block.

47. When 100 grams of alcohol, of density .8, is poured into a cylindrical vessel, it is found to fill it to a depth of 10 centimeters. Find the area of the base of the cylinder in square centimeters.

48. A cylindrical iron bar, 2 centimeters in diameter, has a mass of 3 kilograms. Find the length of the bar.

Let $\pi = 3\frac{1}{7}$.

49. When a body is weighed under water, it is found to be buoyed up by a force equal to the weight of the water which it displaces.

If a boy can exert a lifting force of 120 pounds, how heavy a stone can he lift to the surface of a pond, if the density of stone is 2.5 and that of water 1?

50. When a straight bar is supported at some point, o, and masses m_1, m_2, etc., are hung from the bar as indicated in the figure, it is found that when the bar is in equilibrium, the following relation always holds,

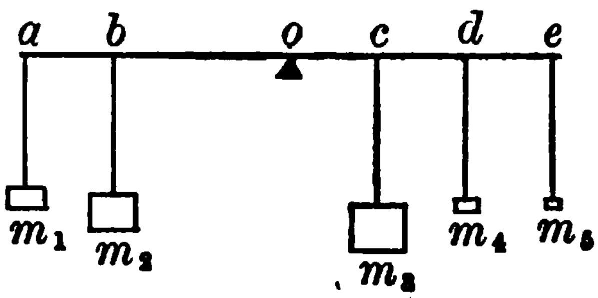


Fig. 1.

$$m_1 \cdot ao + m_2 \cdot bo = m_3 \cdot co + m_4 \cdot do + m_5 \cdot eo.$$

If a teeter board is 10 feet long, where must the support be placed in order that a 70-pound boy at one end may balance a 60-pound boy on the other end plus a 40-pound boy 3 feet from the other end?

51. A bar 40 inches long is in equilibrium when weights of 6 pounds and 9 pounds hang from its two ends. Find the position of the support.

52. If in Fig. 1, $ao=100$, $bo=40$, $co=30$, $do=60$, $eo=110$, and if $m_1=40$, $m_2=60$, $m_3=60$, $m_4=15$, and $m_5=5$, where

must a mass of 100 be placed in order to produce equilibrium?

53. A gas expands $\frac{1}{273}$ of its volume at 0° centigrade for each degree of rise in its temperature; *i. e.*, the volume V_t, at any temperature, t, is connected with the volume V_0, at the temperature 0° centigrade by the equation

$$V_t = V_0 + \tfrac{1}{273}V_0 t,$$

or

$$V_t = V_0(1 + \tfrac{1}{273}t).$$

To what volume will 100 cubic centimeters of air at 0° expand when the temperature rises to 50° centigrade?

54. To what volume will 100 cubic centimeters of air at 50° centigrade contract when the temperature falls to 0° centigrade?

55. To what volume will 100 cubic centimeters of air at 50° expand when the temperature changes to 75°?

56. When a body in motion collides with a body at rest, the momentum of the first body (*i. e.*, the product of its mass, m_1, by its original velocity, v_1) is found to be in every case exactly equal to the total momentum of the two bodies after collision (*i. e.*, to the product of the mass, m_2, of the second body times the velocity, v_2, which it acquires, plus the product of m_1 by the velocity, v_3, which it retains after the collision). The algebraic statement of this relation is

$$m_1 v_1 = m_2 v_2 + m_1 v_3.$$

A billiard ball, the mass of which is 50 grams, and which was moving at a velocity of 1500 centimeters a second, collided with another ball at rest which weighed 30 grams. In the collision the first ball imparted to the second a velocity of 1600 centimeters per second. Find the velocity of the first ball after the collision.

PROBLEMS INVOLVING LITERAL EQUATIONS

150. *Prob.* Divide a into two parts such that m times the first shall exceed n times the second by b.

Let $x=$ one part.

Then, $a-x=$ the other part.

By the conditions, $mx=n(a-x)+b.$

$$mx=an-nx+b,$$

$$mx+nx=an+b,$$

$$x(m+n)=an+b.$$

Whence, $x=\dfrac{an+b}{m+n}$, the first part. (1)

And, $a-x=a-\dfrac{an+b}{m+n}=\dfrac{am+an-an-b}{m+n}$

$=\dfrac{am-b}{m+n}$, the other part. (2)

The results can be used as FORMULÆ *for solving any problem of the above form.*

Thus, let it be required to divide 25 into two parts such that 4 times the first shall exceed 3 times the second by 37.

Here, $a=25$, $m=4$, $n=3$, and $b=37$.

Substituting these values in (1) and (2),

the first part $=\dfrac{25\times3+37}{7}=\dfrac{75+37}{7}=\dfrac{112}{7}=16,$

and the second part $=\dfrac{25\times4-37}{7}=\dfrac{100-37}{7}=\dfrac{63}{7}=9.$

EXERCISE 60

1. Divide a into two parts whose quotient shall be m.

2. If A can do a piece of work in m hours, and A and B together in n hours, in how many hours can B alone do the work?

3. Divide a into two parts such that the sum of one-mth the first and one-nth the second shall equal b.

4. A courier who travels a miles a day is followed by another who travels b miles a day. How many days must the second start after the first to overtake him after c days?

5. Divide a into three parts such that the first shall be one-mth the second and one-nth the third.

6. The length of a field is m times its width. If the length were increased by a feet, and the width by b feet, the area would be increased by c square feet. Find the dimensions of the field.

7. A courier who travels a miles a day is followed after b days by another. How many miles a day must the second courier travel to overtake the first after c days?

8. If A can do a piece of work in a hours, B in b hours, C in c hours, and D in d hours, how many hours will it take to do the work if all work together?

XI. SIMULTANEOUS LINEAR EQUATIONS

CONTAINING TWO OR MORE UNKNOWN NUMBERS

151. An equation containing two or more unknown numbers is satisfied by an unlimited number of sets of values of these numbers.

Consider, for example, the equation $x+y=5$.
Putting $x=1$, we have $1+y=5$, or $y=4$.
Putting $x=2$, we have $2+y=5$, or $y=3$; etc.
Thus the equation is satisfied by the sets of values

$$x=1,\ y=4,$$

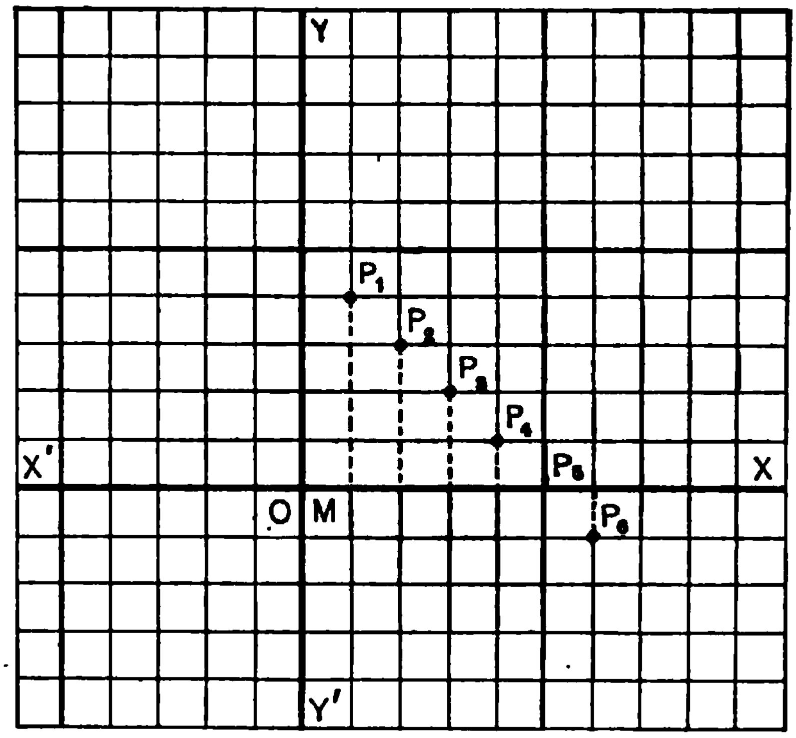
and

$$x=2,\ y=3;\ \text{etc.}$$

An algebraic equation which is satisfied by an unlimited number of such sets of values, is called an **Indeterminate Equation.**

If we agree, as in example 28, Exercise 41, that distances measured toward the right from a definite line and upward from another definite line shall be positive and that measurements in the opposite directions be negative, and also that the vertical measurements shall be *y measurements* and the horizontal distances *x measurements*, a definite picture of the equation $x+y=5$ may be drawn. On *square ruled* paper, choose a horizontal and a vertical line, $X'X$ and $Y'Y$; these lines are called the *x-axis* and *y-axis* respectively.

When $x=1, y=4$, laying off $OM=1$, and $MP_1=4$, we locate the point P_1. *1* and *4* are called the *coördinates* of the point P_1. MP_1 is the *ordinate* and OM the *abscissa* of the point P_1. O is the *origin*. Taking other pairs of values of x and y which satisfy $x+y=5$, we may locate the points P_2, P_3, etc., obtained from $x=2$, $y=3$; $x=3, y=2$; $x=4, y=1$; $x=5, y=0$; $x=6, y=-1$; etc. Connect these points.



The line thus drawn is called the *graph* of the given equation. The graph of an equation of the first degree in x and y (§ 75), is a straight line. Therefore the equation is called *linear*.

Coördinates are often written thus, (x, y), the x coördinate being written first.

EXERCISE 61

1. Locate the points $(2, 5)$; $(3, -2)$; $(-3, 2)$; $(-5, -1)$; $(2, 7)$; $(-9, 4)$.
2. Construct the graph of: $x-y=5$.
3. Construct the graph of: $2x+y=8$.

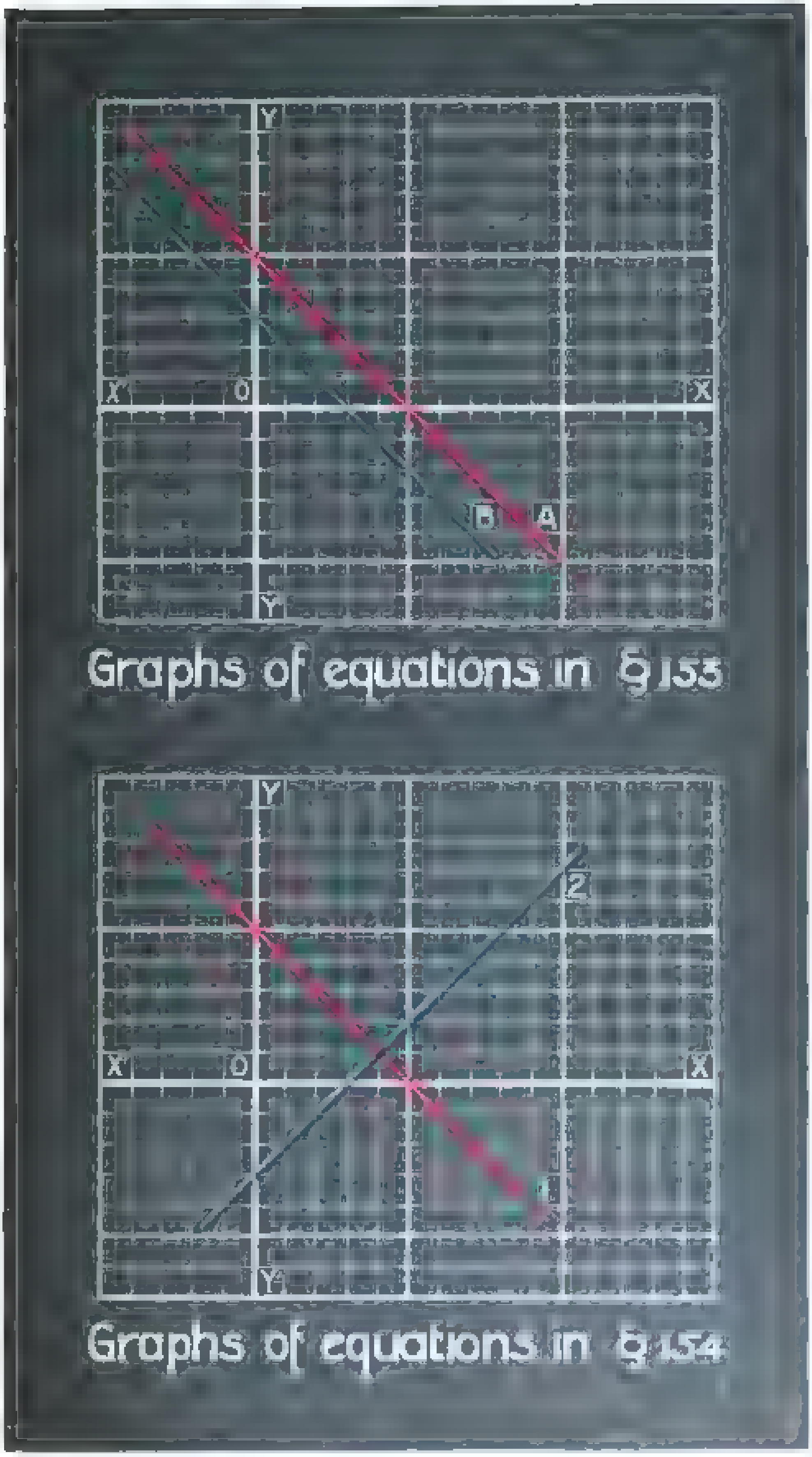
152. Consider the equations

$$\begin{cases} x+y=5, & (1)\\ 2x+2y=10. & (2)\end{cases}$$

Equation (1) can be made to take the form of (2) by multiplying both members by 2; then, every set of values of x and y which satisfies one of the equations also satisfies the other. Such equations are called **Equivalent**.

Again, consider the equations

$$\begin{cases} x+y=5, & (3)\\ x-y=3. & (4)\end{cases}$$



Y
X
O
X
B
A
Y
Graphs of equations in §153
Y
2
X
O
X
Y
Graphs of equations in §154

PLATE I

In this case, it is not true that every set of values of x and y which satisfies one of the equations also satisfies the other; thus, equation (3) is satisfied by the set of values $x=3$, $y=2$, which does not satisfy (4).

If two equations, containing two or more unknown numbers, are not equivalent, they are called **Independent.**

153. Consider the equations

$$\begin{cases} x+y=5, & \text{(A)} \\ x+y=3. & \text{(B)} \end{cases}$$

It is evidently impossible to find a set of values of x and y which shall satisfy both (A) and (B). Such equations are called **Inconsistent.**

154. A system of equations is called **Simultaneous** when each contains two or more unknown numbers, and every equation of the system is satisfied by the same set, or sets, of values of the unknown numbers; thus, each equation of the system

$$\begin{cases} x+y=5, & (1) \\ x-y=3, & (2) \end{cases}$$

is satisfied by the set of values $x=4$, $y=1$.

In § 151, we found that one equation containing two unknown quantities was *indeterminate*. Notice that the *graphs*, Plate I, of $x+y=5$ (1) and $x-y=3$ (2) do not have the same direction. If constructed on the same diagram these lines will cross at some point. Construct the graphs on the same diagram, *i. e.*, use the same axes for both equations, and you will find that the coördinates of the crossing point are the same as the x and y of the set of values in § 154, *i. e.*, $x=4$, $y=1$. The coördinates of every point on the graph of equation (1) satisfy equation (1), also the coördinates of every point on the graph of equation (2) satisfy equation (2), but only at this point (4, 1) do the same coördinates satisfy both equations, hence the name *simultaneous equations*. In solving simultaneous equations we are simply finding the point where the graphs intersect. The finding of this point by means of graphs is somewhat slow and inaccurate, but *algebra* offers several methods by which the point may be readily found.

A **Solution** of a system of simultaneous equations is a set of values of the unknown numbers which satisfies every equa-

tion of the system; to *solve* a system of simultaneous equations is to find its solutions.

155. Two independent simultaneous equations of the form $ax+by=c$ may be solved by combining them in such a way as to form a single equation containing but *one* unknown number. This operation is called **Elimination**.

ELIMINATION BY ADDITION OR SUBTRACTION

156. 1. Solve the equations $\begin{cases} 5x-3y=19. & (1) \\ 7x+4y=2. & (2) \end{cases}$

Multiplying (1) by 4,	$20x-12y=76.$	(3)
Multiplying (2) by 3,	$\underline{21x+12y=6.}$	(4)
Adding (3) and (4),	$41x=82.$	(5)
Whence,	$x=2.$	(6)
Substituting $x=2$ in (1),	$10-3y=19.$	(7)
Whence,	$-3y=9$, or $y=-3.$	(8)

Check this solution by substituting $x=2$, $y=-3$ in the *given* equations.

The above is an example of elimination by addition.

We speak of *adding* a system of equations when we mean placing the sum of the first members equal to the sum of the second members.

Abbreviations of this kind are frequent in Algebra; thus we speak of *multiplying* an equation when we mean multiplying each term of both of its members.

2. Solve the equations $\begin{cases} 15x+8y=1. & (1) \\ 10x-7y=-24. & (2) \end{cases}$

Multiplying (1) by 2,	$30x+16y=2.$	(3)
Multiplying (2) by 3,	$\underline{30x-21y=-72.}$	(4)
Subtracting (4) from (3),	$37y=74$, and $y=2.$	
Substituting $y=2$ in (1),	$15x+16=1.$	
Whence,	$15x=-15$, and $x=-1.$	

The above is an example of elimination by *subtraction.*

From the above examples, we have the following rule:

If necessary, multiply the given equations by such numbers as will make the coefficients of one of the unknown numbers in the resulting equations of equal absolute value.

Add or subtract the resulting equations according as the coefficients of equal absolute value are of unlike or like sign.

If the coefficients which are to be made of equal absolute value are prime to each other, each may be used as the multiplier for the other equation; but if they are not prime to each other, such multipliers should be used as will produce their lowest common multiple.

Thus, in Ex. 1, to make the coefficients of y of equal absolute value, we multiply (1) by 4 and (2) by 3; but in Ex. 2, to make the coefficients of x of equal absolute value, since the L. C. M. of 10 and 15 is 30, we multiply (1) by 2 and (2) by 3.

EXERCISE 62

Solve by the method of addition or subtraction; verify each result:

1. $\begin{cases} 6x+5y=28. \\ 4x+y=14. \end{cases}$

2. $\begin{cases} x-5y=-21. \\ 3x-8y=-35. \end{cases}$

3. $\begin{cases} 2t-3u=19. \\ 7t+4u=23. \end{cases}$

4. $\begin{cases} 11x-15y=-7. \\ 5y+9x=-23. \end{cases}$

5. $\begin{cases} 3u-2v=8. \\ 4u+3v=5. \end{cases}$

6. $\begin{cases} 8m+6k=9. \\ 12m-9k=8. \end{cases}$

ELIMINATION BY SUBSTITUTION

157. *Ex.* Solve the equations $\begin{cases} 7x-9y=15. & (1) \\ 8y-5x=-17. & (2) \end{cases}$

Transposing $-5x$ in (2), $8y=5x-17.$

Whence, $y=\dfrac{5x-17}{8}.$ (3)

Substituting in (1), $7x-9\left(\dfrac{5x-17}{8}\right)=15.$ (4)

Clearing of fractions, $56x-9(5x-17)=120.$

Or, $56x-45x+153=120.$

Uniting terms, $11x=-33.$

Whence, $x=-3.$ (5)

Substituting $x=-3$ in (3), $y=\dfrac{-15-17}{8}=-4.$ (6)

Verify the result.

From the above example, we have the following rule:

From one of the given equations find the value of one of the unknown numbers in terms of the other, and substitute this value in place of that number in the other equation.

ELIMINATION BY COMPARISON

158. *Ex.* Solve the equations $\begin{cases} 2x-5y=-16. & (1) \\ 3x+7y=5. & (2) \end{cases}$

Transposing $-5y$ in (1), $2x=5y-16.$

Whence, $x=\dfrac{5y-16}{2}.$ (3)

Transposing $7y$ in (2), $3x=5-7y.$

Whence, $x=\dfrac{5-7y}{3}.$ (4)

Equating values of x, $\dfrac{5y-16}{2}=\dfrac{5-7y}{3}.$ (5)

Clearing of fractions, $15y-48=10-14y.$

Transposing, $29y=58.$

Whence,. $y=2.$ (6)

Substituting $y=2$ in (3), $x=\dfrac{10-16}{2}=-3.$ (7)

Verify your result.

From the above example, we have the following rule:

From each of the given equations, find the value of the same unknown number in terms of the other, and place these values equal to each other.

EXERCISE 63

Solve by either substitution or comparison, using the method that seems the easier. Verify each result.

1. $\begin{cases} 3v-5y=-5. \\ 7v-3y=10. \end{cases}$

2. $\begin{cases} x+2y=9. \\ x-9y=7. \end{cases}$

3. $\begin{cases} 4w-l=11. \\ w+l=12. \end{cases}$

4. $\begin{cases} 15t+8u=3. \\ 6t-12u=5. \end{cases}$

5. $\begin{cases} 6z+11y=31. \\ 6y-11z=74. \end{cases}$

6. $\begin{cases} 5q+2r=-4. \\ 6q-11r=-45. \end{cases}$

7. $\begin{cases} 12e - 11f = 19. \\ 12f - 11e = -27. \end{cases}$

8. $\begin{cases} 9u + 6v = -16. \\ 13u + 7v = -22. \end{cases}$

9. $\begin{cases} 8m - 15v = 18. \\ 12m + 6v = -11. \end{cases}$

10. $\begin{cases} 5h + 4k = 34. \\ 8h - 3k = 35. \end{cases}$

EXERCISE 64

Solve the following equations, using any method of elimination you choose, and verify each result:

1. $\begin{cases} x + 2y = -2. \\ 4x - 7y = 37. \end{cases}$

2. $\begin{cases} 11v + 4u = 3. \\ 8v + 9u = -10. \end{cases}$

3. $\begin{cases} 6t - 7w = -12. \\ 10t - 9w = -12. \end{cases}$

4. $\begin{cases} 15r + 8s = -14. \\ 6r + 12s = 1. \end{cases}$

5. $\begin{cases} \dfrac{2x}{3} + \dfrac{3y}{4} = -\dfrac{7}{2}. \\ \dfrac{x}{4} - \dfrac{2y}{5} = \dfrac{11}{2}. \end{cases}$

If the given equations are not in the form $ax + by = c$, they should be reduced to this form before applying any method of elimination.

6. $\begin{cases} 8e + 7f = 12. \\ \dfrac{e+2f}{4} + \dfrac{2e+f}{3} = 1. \end{cases}$

7. $\begin{cases} \dfrac{y}{2} - \dfrac{x}{2} = 2. \\ \dfrac{3-2x}{5} - \dfrac{4+5y}{11} = 4. \end{cases}$

8. $\begin{cases} \dfrac{x+2}{7} + 8 = 2x - \dfrac{y-x}{4}. \\ 3x - \dfrac{2y-3x}{3} = 2y - 4. \end{cases}$

9. $\begin{cases} \dfrac{8-y}{5} - \dfrac{2u+3}{4} = \dfrac{y+3}{4}. \\ \dfrac{1+4y}{11} - \dfrac{u+7}{3} = 3. \end{cases}$

In solving fractional simultaneous equations, we reject any solution which does not satisfy the given equations.

10. Solve the equations $\begin{cases} 2x + 3y = 13. & (1) \\ \dfrac{1}{x-2} + \dfrac{1}{y-3} = 0. & (2) \end{cases}$

Multiplying each term of (2) by $(x-2)(y-3)$, we have

$$y-3+x-2=0, \text{ or } y=-x+5. \qquad (3)$$

Substituting in (1), $2x-3x+15=13$, or $x=2$.

Substituting in (3), $y=-2+5=3$.

This solution satisfies the first given equation, but not the second; then it must be rejected.

11. $\begin{cases} 3x-4y=-11. \\ \dfrac{2}{x+5}-\dfrac{5}{y+1}=0. \end{cases}$

12. $\begin{cases} x-\dfrac{4y-9}{11}=5. \\ \dfrac{9}{2}-\dfrac{x+5}{3}=-3y. \end{cases}$

13. $\begin{cases} (2m-1)(z-4)-(m-5)(2z+5)=121. \\ 4m-3z=-29. \end{cases}$

14. $\begin{cases} \dfrac{7}{u-3}-\dfrac{8}{v-5}=0. \\ \dfrac{9}{2u-1}-\dfrac{5}{3v+4}=0. \end{cases}$

15. $\begin{cases} \dfrac{x+11}{7}+\dfrac{y-6}{5}=-4. \\ \dfrac{x-1}{2}-\dfrac{y+4}{10}=-45. \end{cases}$

16. $\begin{cases} \dfrac{d-2n}{3d+n+3}=-\dfrac{1}{5}. \\ \dfrac{d+3n}{d+4n-7}=\dfrac{7}{11}. \end{cases}$

17. $\begin{cases} .08x+.9y=.048. \\ .3x-.35y=.478. \end{cases}$

18. $\begin{cases} \dfrac{2x-3y}{4}+\dfrac{4x+6y}{3}=-\dfrac{1}{2}. \\ \dfrac{5x+2y}{2}+\dfrac{7y-3x}{5}=\dfrac{39}{10}. \end{cases}$

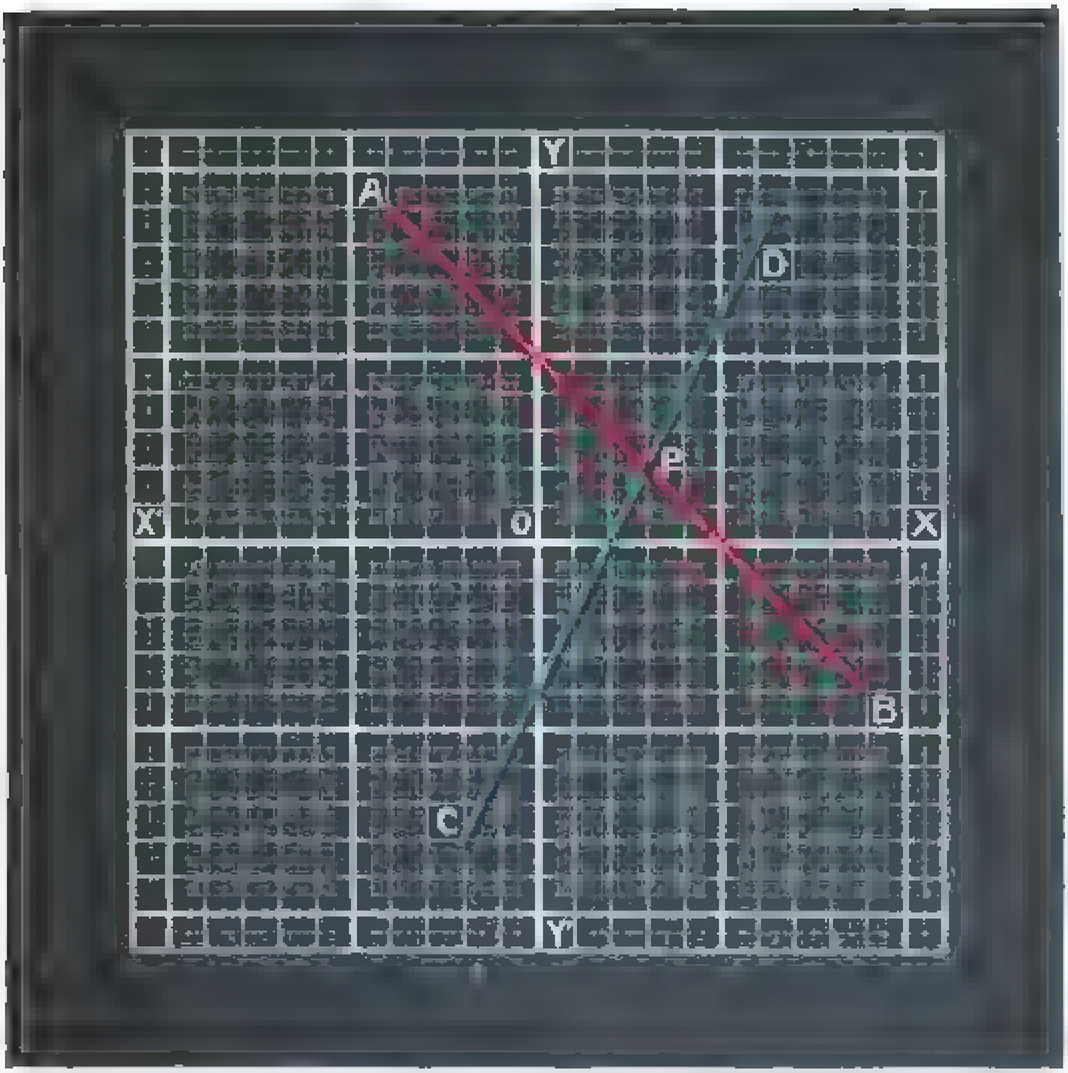
19. $\begin{cases} \dfrac{h+k}{h-k}=-\dfrac{1}{10}. \\ \dfrac{3h+8}{k-4}=\dfrac{6h-1}{2k+3}. \end{cases}$

20. $\begin{cases} 5x-\frac{1}{3}(3x-2y+5)=11. \\ \frac{5}{6}(x-4y)-\frac{4}{9}(x-y)=16. \end{cases}$

21. $\begin{cases} \dfrac{21g+1}{2}-\dfrac{5v+2}{3}=\dfrac{63g-130v}{21}. \\ \dfrac{14g+1}{3}-\dfrac{10v-3}{5}=2. \end{cases}$

Certain equations in which the unknown numbers occur in the denominators of fractions may be readily solved without previously clearing of fractions.

22. $\begin{cases} \dfrac{10}{x}-\dfrac{9}{y}=8. & (1) \\ \dfrac{8}{x}+\dfrac{15}{y}=-1. & (2) \end{cases}$



$x+y=5$ (AB)
$2x-y=4$ (CD)

$x+y=5$

x	y
0	5
1	4
2	3
3	2
4	1
+5	0
−1	6

$2x-y=4$

x	y
0	−4
1	−2
2	0
3	2
4	4
−1	−6
−2	−8

Solving the equations for x and y, we have $x=3$, $y=2$. Note that these values correspond to the x and y coördinates of the point P.

In general, in solving two simultaneous equations in two unknown quantities, the real values found for the unknown quantities correspond to the intersection points of the graphs of the given equations.

PLATE II

Multiplying (1) by 5, $\frac{50}{x} - \frac{45}{y} = 40.$

Multiplying (2) by 3, $\frac{24}{x} + \frac{45}{y} = -3.$

Adding, $\frac{74}{x} = 37,\ 74 = 37x$, and $x = 2.$

Substituting in (1), $5 - \frac{9}{y} = 8,\ -\frac{9}{y} = 3$, and $y = -3.$

23. $\begin{cases} \frac{3}{x} - \frac{3}{y} = -\frac{1}{4}. \\ \frac{6}{x} - \frac{3}{y} = \frac{1}{2}. \end{cases}$

24. $\begin{cases} \frac{6}{t} + \frac{12}{w} = -1. \\ \frac{8}{t} - \frac{9}{w} = 7. \end{cases}$

25. $\begin{cases} \frac{3}{u} - \frac{4}{v} = 1. \\ \frac{4}{u} + \frac{3}{v} = 2. \end{cases}$

26. $\begin{cases} \frac{9}{x} - 3y = 4. \\ \frac{3}{x} + 2y = \frac{10}{3}. \end{cases}$

27. $\begin{cases} \frac{6}{r} - \frac{2}{s} = -7. \\ \frac{4}{r} + \frac{6}{s} = -1. \end{cases}$

28. $\begin{cases} 3m - \frac{5}{k} = 10. \\ 2m + \frac{10}{k} = 4. \end{cases}$

159. In graphical work the drawings are much more effective and pleasing if one uses a color scheme similar to that in Plate II. Cross-ruled paper and color crayons, for either blackboard or paper, can be secured at nominal expense.

EXERCISE 65

Construct the graphs of the following equations, in each case comparing the coördinates of the intersection with your algebraic solution:

1. $\begin{cases} x + y = 8. \\ x - y = 6. \end{cases}$

2. $\begin{cases} 2x + y = 10. \\ x + 2y = 4. \end{cases}$

3. $\begin{cases} 5x - 6y = -9. \\ 3x - 5y = -4. \end{cases}$

4. $\begin{cases} 8x - 3y = 47. \\ 6x - 7y = 21. \end{cases}$

5. $\begin{cases} x + 2y = 10. \\ 2x + 4y = 30. \end{cases}$

6. $\begin{cases} 2x - 3y = 12. \\ 4x - 6y = 24. \end{cases}$

7. $\begin{cases} 3x-4y=10. \\ 4x+3y=\ \ 5. \end{cases}$ 8. $\begin{cases} x-y=5. \\ x-y=8. \end{cases}$

9. $x+y=8$. Find the area of the triangle formed by this line and the axes.

10. $\begin{cases} 2x-\ \ y=\ \ 2. \\ 4x-2y=16. \end{cases}$ 11. $\begin{cases} 3x-\ \ y=-9. \\ 3x+2y=-6. \end{cases}$

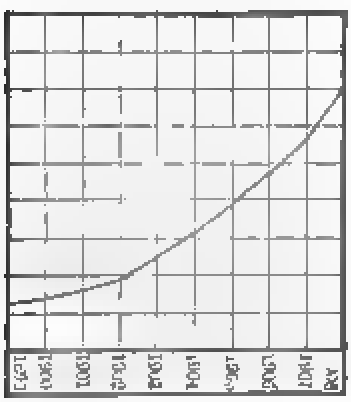
An interesting application of the graph is the construction of the geometric picture of related data:

12. The enrollment of pupils in the Seattle high school for ten years is as follows:

Year	No. of Pupils	Year	No. of Pupils	Year	No. of Pupils
1899	592	1903	1213	1906	2312
1900	684	1904	1522	1907	2794
1901	800	1905	1960	1908	3500
1902	947				

Choosing 1 year for the horizontal unit and 500 for the vertical unit, we have the following graph:

Vertical Scale, 500=1 Unit
Horizontal Scale, 1=1 Unit



Horizontal	*Vertical*
0	592
1	684
2	800
3	947
4	1213
5	1522
6	1960
7	2312
8	2794
9	3500

Construct the graphs of the following:

13. The enrollment in the Toledo high school is

Year	No. of Pupils	Year	No. of Pupils	Year	No. of Pupils
1898	773	1902	1376	1905	1622
1899	984	1903	1414	1906	1791
1900	1110	1904	1500	1907	1900
1901	1102				

14. Standing of the Chicago National League Baseball Team:

Year	Percentage	Year	Percentage	Year	Percentage
1898	.567	1902	.497	1905	.601
1899	.507	1903	.594	1906	.763
1900	.474	1904	.608	1907	.704
1901	.381				

15. Enrollment in Cleveland high schools:

Year	No. of Pupils	Year	No. of Pupils	Year	No. of Pupils
1898	3378	1901	3595	1904	4491
1899	3460	1902	3796	1905	5001
1900	3589	1903	4151	1906	5070

16. Enrollment in Chicago high schools:

Year	No. of Pupils	Year	No. of Pupils	Year	No. of Pupils
1897	7847	1901	9661	1904	9936
1898	8432	1902	9627	1905	11208
1899	8830	1903	9488	1906	12024
1900	9190				

17. Standing of the Detroit American League Baseball Team:

Year	Percentage	Year	Percentage	Year	Percentage
1901	.548	1904	.408	1906	.477
1902	.385	1905	.516	1907	.613
1903	.478				

18. Enrollment in New York City high schools:

Year	No. of Pupils	Year	No. of Pupils	Year	No. of Pupils
1899	13731	1902	21461	1905	30340
1900	17018	1903	23701	1906	31949
1901	19013	1904	27794		

160. Solution of Literal Simultaneous Equations. — In solving *literal* simultaneous linear equations, the method of elimination by addition or subtraction is usually to be preferred.

Ex. Solve the equations $\begin{cases} ax + by = c. & (1) \\ a'x + b'y = c'. & (2) \end{cases}$

Multiplying (1) by b', $\quad ab'x + bb'y = b'c.$

Multiplying (2) by b, $\quad a'bx + bb'y = bc'.$

Subtracting $\quad (ab' - a'b)x = b'c - bc'.$

Whence, $x=\dfrac{b'c-bc'}{ab'-a'b}.$

Multiplying (1) by a', $aa'x+a'by=ca'.$ (3)

Multiplying (2) by a, $aa'x+ab'y=c'a.$ (4)

Subtracting (3) from (4), $(ab'-a'b)y=c'a-ca'.$

Whence, $y=\dfrac{c'a-ca'}{ab'-a'b}.$

In solving *fractional* literal simultaneous equations, any solution which does not satisfy the given equations must be rejected. (Compare Ex. 10, Exercise 64.)

EXERCISE 66

Solve the following:

1. $\begin{cases} 5x-6y=8a. \\ 4x+9y=7a. \end{cases}$

2. $\begin{cases} ax+by=1. \\ cx+dy=1. \end{cases}$

3. $\begin{cases} a_1x+a_2y=b_1. \\ a_2x-a_1y=b_2. \end{cases}$

4. $\begin{cases} mx-ny=mn. \\ m'x+n'y=m'n'. \end{cases}$

5. $\begin{cases} \dfrac{2ax-by}{a}=b. \\ \dfrac{x+by}{3a+2}=b. \end{cases}$

6. $\begin{cases} \dfrac{x}{m_1}+\dfrac{y}{m_2}=\dfrac{1}{m_3}. \\ \dfrac{x}{n_1}+\dfrac{y}{n_2}=\dfrac{1}{n_3}. \end{cases}$

7. $\begin{cases} bx-ay=b^2. \\ (a-b)x+by=a^2. \end{cases}$

8. $\begin{cases} \dfrac{m}{n+y}=\dfrac{n}{m-x}. \\ \dfrac{m}{n+x}=\dfrac{n}{m-y}. \end{cases}$

9. $\begin{cases} ax+by=2a. \\ a^2x-b^2y=a^2+b^2. \end{cases}$

10. $\begin{cases} (a+1)x+(a-2)y=3a. \\ (a+3)x+(a-4)y=7a. \end{cases}$

11. $\begin{cases} ab(a-b)x+ab(a+b)y=a^2+2ab-b^2. \\ ax+by=2. \end{cases}$

12. $\begin{cases} \dfrac{a}{x}+\dfrac{b}{y}=c. \\ \dfrac{a'}{x}+\dfrac{b'}{y}=c'. \end{cases}$

13. $\begin{cases} \dfrac{a}{bx}+\dfrac{b}{ay}=\dfrac{a+b}{ab}. \\ \dfrac{b}{ax}-\dfrac{a}{by}=\dfrac{b^3-a^3}{a^2b^2}. \end{cases}$

14. $\begin{cases} m(x+y)+n(x-y)=2. \\ m^2(x+y)-n^2(x-y)=m-n. \end{cases}$

15. $\begin{cases} (a+b)x+(a-b)y=2(a^2+b^2). \\ \dfrac{b}{x-a-b}=\dfrac{a}{y-a+b}. \end{cases}$

16. $\begin{cases} (a+b)x+(a-b)y=2a^2-2b^2. \\ \dfrac{y}{a-b}-\dfrac{x}{a+b}=\dfrac{4ab}{a^2-b^2}. \end{cases}$

17. $\begin{cases} bx+ay=2. \\ ab(a+b)x-ab(a-b)y=a^2+b^2. \end{cases}$

SIMULTANEOUS EQUATIONS CONTAINING MORE THAN TWO UNKNOWN NUMBERS

161. If we have *three* independent simultaneous equations, containing *three* unknown numbers, we may combine any two of them by one of the methods of elimination explained in §§ 156 to 158, so as to obtain a single equation containing only two unknown numbers.

We may then combine the remaining equation with either of the other two, and obtain another equation containing the *same* two unknown numbers.

By solving the two equations containing two unknown numbers, we may obtain their values; and substituting them in either of the given equations, the value of the remaining unknown number may be found.

We proceed in a similar manner when the number of equations and of unknown numbers is greater than three.

The method of elimination by addition or subtraction is usually the most convenient.

In solving *fractional* simultaneous equations, any solution which does not satisfy the given equations must be rejected. (Ex. 10, Exercise 64.)

1. Solve the equations $\begin{cases} 6x-4y-7z=17. & (1) \\ 9x-7y-16z=29. & (2) \\ 10x-5y-3z=23. & (3) \end{cases}$

Multiplying (1) by 3,	$18x-12y-21z=51.$	
Multiplying (2) by 2,	$18x-14y-32z=58.$	
Subtracting,	$2y+11z=-7.$	(4)
Multiplying (1) by 5,	$30x-20y-35z=85.$	(5)
Multiplying (3) by 3,	$30x-15y-9z=69.$	(6)
Subtracting (5) from (6)	$5y+26z=-16.$	(7)

Multiplying (4) by 5,	$10y+55z=-35.$
Multiplying (7) by 2,	$\underline{10y+52z=-32.}$
Subtracting,	$3z=-3$, or $z=-1.$
Substituting in (7),	$2y-11=-7$, or $y=2.$
Substituting in (1),	$6x-8+7=17$, or $x=3.$

EXERCISE 67

Solve the following:

*1. $\begin{cases} 4x-3y=-5. \\ 4y-3z=-13. \\ 4z-3x=18. \end{cases}$

2. $\begin{cases} 4x-5y-6z=0. \\ x-y+z=1. \\ 9x+z=8. \end{cases}$

3. $\begin{cases} 3x+y-z=14. \\ x+3y-z=16. \\ x+y-3z=-10. \end{cases}$

4. $\begin{cases} g+h-k=24. \\ 4g+3h-k=61. \\ 6g-5h-k=11. \end{cases}$

5. $\begin{cases} 3x+5y=5. \\ 9x+5z=55. \\ 9y+3z=-30. \end{cases}$

6. $\begin{cases} 5m-y+4v=-5. \\ 3m+5y+6v=-20. \\ m+3y-8v=-27. \end{cases}$

7. $\begin{cases} 2x-5y=-26. \\ 7x+6z=-33. \\ \dfrac{3}{y-4}=\dfrac{4}{z+2}. \end{cases}$

8. $\begin{cases} 2x+4y-z=-2. \\ 18x-8y+4z=-25. \\ 10x+4y-9z=-30. \end{cases}$

9. $\begin{cases} 3p+4q+5r=10. \\ 4p-5q-3r=25. \\ 5p-3q-4r=21. \end{cases}$

10. $\begin{cases} 4u-11v-5w=9. \\ 8u+4v-w=11. \\ 16u+7v+6w=64. \end{cases}$

11. $\begin{cases} 8x+4y+3z=-52. \\ 5x-y+12z=-52. \\ 9x+7y-6z=-36. \end{cases}$

12. $\begin{cases} 6r-s+3t=42. \\ 10r-5s-t=2. \\ 6r-17s+4t=-46. \end{cases}$

13. $\begin{cases} 2x+5y+3z=-7. \\ 2y-4z=2-3x. \\ 5x+9y=5+7z. \end{cases}$

14. $\begin{cases} \dfrac{5}{x}-\dfrac{8}{y}=-3. \\ \dfrac{8}{y}-\dfrac{3}{z}=1. \\ \dfrac{25}{z}+\dfrac{7}{3x}=2. \end{cases}$

* Eliminate y from (1) and (2) you then have two equations in x and z; or add the three given equations.

15. $\begin{cases} \dfrac{2}{3x}+\dfrac{1}{y}=-\dfrac{3}{10}. \\ \dfrac{3}{4y}-\dfrac{1}{z}=\dfrac{7}{30}. \\ \dfrac{4}{5z}+\dfrac{1}{x}=\dfrac{1}{12}. \end{cases}$

16. $\begin{cases} ax+by=\dfrac{a^3+b^3}{abc}. \\ by+cz=\dfrac{b^3+c^3}{abc}. \\ cz+ax=\dfrac{c^3+a^3}{abc}. \end{cases}$

PROBLEMS INVOLVING SIMULTANEOUS EQUATIONS WITH TWO OR MORE UNKNOWN NUMBERS

162. In solving problems where two or more letters are used to represent unknown numbers, we must obtain from the conditions of the problem *as many independent equations* (§ 152) *as there are unknown numbers to be determined.*

1. Divide 81 into two parts such that three-fifths the greater shall exceed five-ninths the less by 7.

Let $x=$ the greater part,
and $y=$ the less.

By the conditions, $x+y=81$, (1)

and $\dfrac{3x}{5}=\dfrac{5y}{9}+7.$ (2)

Solving (1) and (2), $x=45$, $y=36$.

2. If 3 be added to both numerator and denominator of a fraction, its value is $\frac{2}{3}$; and if 2 be subtracted from both numerator and denominator, its value is $\frac{1}{2}$; find the fraction.

Let $n=$ the numerator,
and $d=$ the denominator.

By the conditions, $\dfrac{n+3}{d+3}=\dfrac{2}{3}$,

and $\dfrac{n-2}{d-2}=\dfrac{1}{2}.$

Solving these equations, $n=7$, $d=12$; then, the fraction is $\dfrac{7}{12}$.

3. A sum of money was divided equally among a certain number of persons. Had there been 3 more, each would have received \$1 less; had there been 6 fewer, each would have received \$5 more. How many persons were there, and how much did each receive?

Let $x =$ the number of persons,
and $y =$ the number of dollars received by each.
Then, $xy =$ the number of dollars divided.

Since the sum of money could be divided between $x+3$ persons, each of whom would receive $y-1$ dollars, and between $x-6$ persons, each of whom would receive $y+5$ dollars, $(x+3)(y-1)$ and $(x-6)(y+5)$ also represent the number of dollars divided.

Then, $(x+3)(y-1) = xy,$
and $(x-6)(y+5) = xy.$
Solving these equations, $x = 12,\ y = 5.$

4. The sum of the three digits of a number is 13. If the number, decreased by 8, be divided by the sum of its second and third digits, the quotient is 25; and if 99 be added to the number, the digits will be inverted. Find the number.

Let $x =$ the first digit,
$y =$ the second,
and $z =$ the third.
Then, $100x + 10y + z =$ the number,
and $100z + 10y + x =$ the number with its digits inverted.

By the conditions of the problem,

$$x + y + z = 13,$$

$$\frac{100x + 10y + z - 8}{y+z} = 25,$$

and $$100x + 10y + z + 99 = 100z + 10y + x.$$

Solving these equations, $x = 2,\ y = 8,\ z = 3$; and the number is 283.

5. A crew can row 10 miles in 50 minutes down stream, and 12 miles in $1\frac{1}{2}$ hours against the stream. Find the rate in miles per hour of the current, and of the crew in still water.

Let $x =$ number of miles an hour of the crew in still water,
and $y =$ number of miles an hour of the current.
Then, $x + y =$ number of miles an hour of the crew down stream,
and $x - y =$ number of miles an hour of the crew up stream.

The number of miles an hour rowed by the crew is equal to the distance in miles divided by the time in hours.

Then, $$x + y = 10 \div \frac{5}{6} = 12,$$

and $$x - y = 12 \div \frac{3}{2} = 8.$$

Solving these equations, $x = 10,\quad y = 2.$

6. A train running from A to B meets with an accident which causes its speed to be reduced to one-third of what it was before, and it is in consequence 5 hours late. If the accident had happened 60 miles nearer B, the train would have been only 1 hour late. Find the rate of the train before the accident, and the distance to B from the point of detention.

Let $3x$ = the number of miles an hour of the train before the accident.
Then, x = the number of miles an hour after the accident.
Let y = the number of miles to B from the point of detention.

The train would have done the last y miles of its journey in $\frac{y}{3x}$ hours; but owing to the accident, it does the distance in $\frac{y}{x}$ hours.

Then, $$\frac{y}{x} = \frac{y}{3x} + 5. \tag{1}$$

If the accident had occurred 60 miles nearer B, the distance to B from the point of detention would have been $y - 60$ miles.

Had there been no accident, the train would have done this in $\frac{y-60}{3x}$ hours, and the accident would have made the time $\frac{y-60}{x}$ hours.

Then, $$\frac{y-60}{x} = \frac{y-60}{3x} + 1. \tag{2}$$

Subtracting (2) from (1), $\frac{60}{x} = \frac{60}{3x} + 4$, or $\frac{40}{x} = 4$; whence, $x = 10$.

Then, the rate of the train before the accident was 30 miles an hour.

Substituting in (1), $\frac{y}{10} = \frac{y}{30} + 5$, or $\frac{y}{15} = 5$; whence, $y = 75$.

EXERCISE 68

1. If the numerator of a fraction be decreased by 1, the value of the fraction is $\frac{1}{3}$, while if 7 be added to both numerator and denominator, the value of the fraction is $\frac{11}{16}$; find the fraction.

2. The sum of two numbers is 7. The ratio of their product to the product of three times the first number and the second increased by 2 is $\frac{1}{5}$. What are the numbers?

3. The sum of the two digits of a number is 14; and if 36 be added to the number, the digits will be reversed. Find the number.

4. Nine shares of N. Y. Central stock and 7 shares of Illinois Central stock cost \$1702, and 5 shares of I. C. cost \$35 more than 6 shares of N. Y. C. Find the cost of one share of each.

5. Find two numbers such that the ratio of the first number to itself increased by 3 is equal to the ratio of the second number to itself increased by $\frac{5}{8}$; and the sum of the two numbers is to twice their difference as 7 is to 4.

6. If 3 be added to the numerator of a fraction, and 7 subtracted from the denominator, its value is $\frac{6}{7}$; and if 1 be subtracted from the numerator, and 7 added to the denominator, its value is $\frac{2}{5}$. Find the fraction.

7. Find two numbers such that one shall be n times as much greater than a as the other is less than a; and the quotient of their sum by their difference equal to b.

8. A wheat field is 80 rods longer than it is wide, and the distance around the field is $1\frac{1}{2}$ miles. Find the length and breadth.

9. In plowing the long way of the above field, a farmer finds he can turn 33 twelve-inch furrows a day. At \$3.50 per day for a man and team, what is the cost of plowing per acre?

10. C's age is three times the sum of A's and B's. Three times B's age added to A's is 12 years less than C's, and if 8 years be subtracted from C's age and this difference be divided by B's age, the quotient will be 4. Find their ages.

11. A rectangular mirror is 6 inches longer than it is wide. It is surrounded by a frame 3 inches wide, whose area is 216 square inches. How much wall space will the mirror and frame occupy?

12. A man had \$3000 in a savings bank which paid him 3% interest. He drew out a part of his money and invested it in municipal bonds which paid him 5%. His annual

income from the entire sum was then \$126. Find the amount left in the savings account.

13. If we consider $y=kx$ a proportion, what is the ratio of y to x? Give k some definite value and make the graph of the equation. If perpendiculars are dropped from the graph to the x-axis, triangles are formed by the perpendiculars, the graph, and the x-axis. Are the triangles alike in form? Is the ratio of the altitude to the base the same in each? If k is given a different value from the one chosen, what effect does this have on the graph and on the triangles?

14. A rectangular field has the same area as another which is 6 rods longer and 2 rods narrower, and also the same area as a third which is 3 rods shorter and 2 rods wider. Find its dimensions.

15. Find three numbers such that the first with one-half the second and one-third the third shall equal 29; the second with one-third the first and one-fourth the third shall equal 28; and the third with one-half the first and one-third the second shall equal 36.

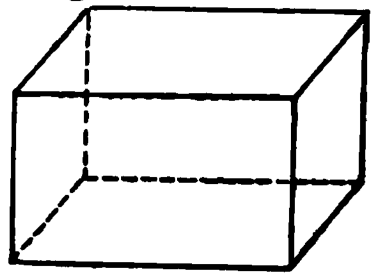
16. The circumference of the large wheel of a carriage is 55 inches more than that of the small wheel. The former makes as many revolutions in going 250 feet as the latter does in going 140 feet. Find the number of inches in the circumference of each wheel.

17. A man having \$4500 in a savings bank which paid him 3 % interest withdrew the money, investing a part in Rock Island 5 % bonds for which he paid \$80 (par value 100); with the balance he purchased Pennsylvania Railway 5 % bonds at par. His annual income from these investments was \$255. Find the amount invested in Rock Island bonds, and their face value.

18. A number consists of two digits. If the first digit be divided by one less than the second digit, the quotient is 3.

If the first digit, increased by 3, be divided by the second digit the quotient is 3; find the number. State your problem, then compare § 152.

19. The sum of the length, breadth, and height of a rectangular parallelopiped is 20. The difference between the length and height is $\frac{2}{5}$ the sum of the height and breadth, and three times the breadth added to the height is 8 more than the length; find the dimensions.



20. If the digits of a number of three figures be reversed, the sum of the number thus formed and the original number is 1615; the sum of the digits is 20, and if 99 be added to the number, the digits will be inverted. Find the number.

21. A train left A for B, 112 miles distant, at 9 A. M., and one hour later a train left B for A; they met at 12 noon. If the second train had started at 9 A. M., and the first at 9.50 A. M., they would also have met at noon. Find their rates.

22. A boy has $1.50 with which he wishes to buy two kinds of note-books. If he asks for 14 of the first kind, and 11 of the second, he will require 6 cents more; and if he asks for 11 of the first kind, and 14 of the second, he will have 6 cents over. How much does each kind cost?

23. The difference between the length and breadth of a rectangle is 6. If the length were diminished by 3 feet and the breadth increased by 3 feet, the area would be increased by 9 square feet and the figure would be a square; find the dimensions.

Have you more conditions than you need? Are your conditions independent? (Compare § 83.)

24. A number consisting of two digits is such that if the digits be reversed the number formed is 27 less than the original number. The product of the digits is to their difference as the second digit is to $\frac{3}{8}$; find the number.

25. A man invests \$10,000, part at $4\frac{1}{4}\%$, and the rest at $3\frac{1}{2}\%$. He finds that six years' interest on the first investment exceeds five years' interest on the second by \$658. How much does he invest at each rate?

26. A man buys apples, some at 2 for 3 cents, and others at 3 for 2 cents, spending in all 80 cents. If he had bought $\frac{5}{4}$ as many of the first kind, and $\frac{6}{5}$ as many of the second, he would have spent 99 cents. How many of each kind did he buy?

27. An annual income of \$800 is obtained in part from money invested at $3\frac{1}{2}\%$, and in part from money invested at 3%. If the amount invested at the first rate were invested at 3%, and the amount invested at the second rate were invested at $3\frac{1}{2}\%$, the annual income would be \$825. How much is invested at each rate?

28. The contents of one barrel is $\frac{5}{6}$ wine, and of another $\frac{8}{9}$ wine. How many gallons must be taken from each to fill a barrel whose capacity is 24 gallons, so that the mixture may be $\frac{7}{8}$ wine?

29. A boy spends his money for oranges. Had he bought m more, each would have cost a cents less; if n fewer, each would have cost b cents more. How many did he buy, and at what price?

30. A vessel contains a mixture of wine and water. If 50 gallons of wine are added, there is $\frac{7}{8}$ as much wine as water; if 50 gallons of water are added, there is 4 times as much water as wine. Find the number of gallons of wine and water at first.

31. A man buys 15 bottles of sherry, and 20 bottles of claret, for \$38. If the sherry had cost $\frac{5}{4}$ as much, and the claret $\frac{4}{5}$ as much, the wine would have cost \$38.50. Find the cost per bottle of the sherry, and of the claret.

32. If a field were made a feet longer, and b feet wider, its area would be increased by m square feet; but if its length

were made c feet less, and its width d feet less, its area would be decreased by n square feet. Find its dimensions.

33. If the numerator of a fraction be increased by a, and the denominator by b, the value of the fraction is $\frac{m}{n}$; and if the numerator be decreased by c, and the denominator by d, the value of the fraction is $\frac{n}{m}$. Find the numerator and denominator.

34. A certain number equals 59 times the sum of its three digits. The sum of the digits exceeds twice the ten's digit by 3; and the sum of the hundred's and ten's digits exceeds twice the unit's digit by 6. Find the number.

35. A piece of work can be done by A and B in $4\frac{4}{5}$ hours, by B and C in $2\frac{2}{3}$ hours, and by A and C in 3 hours. In how many hours can each alone do the work?

36. The numerator of a fraction has the same two digits as the denominator, but in reversed order; the denominator exceeds the numerator by 9, and if 1 be added to the numerator the value of the fraction is $\frac{3}{4}$. Find the fraction.

37. A man walks from one place to another in $5\frac{1}{2}$ hours. If he had walked $\frac{1}{4}$ of a mile an hour faster, the walk would have taken $36\frac{2}{3}$ fewer minutes. How many miles did he walk, and at what rate?

38. A man invests a certain sum of money at a certain rate of interest. If the principal had been \$1200 greater, and the rate 1% greater, his income would have been increased by \$118. If the principal had been \$3200 greater, and the rate 2% greater, his income would have been increased by \$312. What sum did he invest, and at what rate?

39. A crew row $16\frac{1}{2}$ miles up stream and 18 miles down stream in 9 hours. They then row 21 miles up stream and $19\frac{1}{2}$ miles down stream in 11 hours. Find the rate in miles an hour of the stream, and of the crew in still water.

40. A man buys a certain number of \$100 railway shares, when at a certain rate per cent discount, for \$1050; and when at a rate per cent premium twice as great, sells one-half of them for \$1200. How many shares did he buy, and at what cost?

163. Interpretation of Solutions.

1. The length of a field is 10 rods, and its breadth 8 rods; how many rods must be added to the breadth so that the area may be 60 square rods?

Let x = number of rods to be added.

By the conditions, $10(8+x)=60.$

Then, $80+10x=60$, or $x=-2.$

This signifies that 2 rods must be *subtracted* from the breadth in order that the area may be 60 square rods. (Compare § 11.)

If we should modify the problem so as to read:

"The length of a field is 10 rods, and its breadth 8 rods; how many rods must be *subtracted* from the breadth so that the area may be 60 square rods?"

and let x denote the number of rods to be subtracted, we should find $x=2.$

A negative result sometimes indicates that the problem is impossible. It sometimes indicates that measurement is taken in an opposite direction (Ex. 28, Exercise 41).

2. If 11 times the number of persons in a certain house, increased by 18, be divided by 4, the result equals twice the number increased by 3; find the number.

Let x = the number.

By the conditions, $\frac{11x+18}{4}=2x+3.$

Whence, $11x+18=8x+12$, and $x=-2.$

The negative result shows that the problem is impossible.

A problem may also be impossible when the solution is fractional.

XII. INVOLUTION AND EVOLUTION

164. Involution is the process of raising an expression to any power whose exponent is a positive integer.

We gave in § 88 a rule for raising a monomial to any power whose exponent is a positive integer.

165. If an expression when raised to the nth power, n being a positive integer, is equal to another expression, the first expression is said to be the nth **Root** of the second.

Thus, if $a^n = b$, a is the nth root of b;

if $5^2 = 25$, 5 is the square root of 25.

Evolution is the process of finding any required root of an expression.

166. The **Radical Sign**, $\sqrt{\ }$, when written before an expression, indicates some root of the expression.

Thus, $\sqrt{a}$ indicates the *second*, or *square* root of a;

$\sqrt[3]{a}$ indicates the *third*, or *cube* root of a;

$\sqrt[4]{a}$ indicates the *fourth* root of a; and so on.

The **Index** of a root is the number written over the radical sign to indicate what root of the expression is taken.

If no index is expressed, the index 2 is understood.

An *even* root is one whose index is an even number; an *odd* root is one whose index is an odd number.

167. A Power of a Fraction.

We have, $$\left(\frac{a}{b}\right)^3 = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a \times a \times a}{b \times b \times b} = \frac{a^3}{b^3};$$

and a similar result holds for any positive integral power of $\frac{a}{b}$.

Then, a fraction may be raised to any power whose exponent is a positive integer by raising both numerator and denominator to the required power.

Ex. $\left(-\frac{2x^4}{3y^3}\right)^5 = -\left(\frac{2x^4}{3y^3}\right)^5$ (§ 49) $= -\frac{(2x^4)^5}{(3y^3)^5} = -\frac{32x^{20}}{243y^{15}}$ (§ 88).

EXERCISE 69

Find the values of the following:

1. $\left(\frac{6a^9b^6}{7c^8d^7}\right)^2$.
3. $\left(-\frac{3a^2x^7}{b^8y}\right)^5$.
5. $\left(-\frac{2m^7x^2}{n^5y^4}\right)^7$.
2. $\left(\frac{9mn^4}{8p^5}\right)^3$.
4. $\left(-\frac{4x^m}{5y^4z^9}\right)^4$.
6. $\left(\frac{a^3m^5}{3b^8cn^6}\right)^6$.

168. A Root of a Monomial. To find any root of a monomial which is a perfect power of the same degree as the index of the required root.

1. Required the cube root of $a^3b^9c^6$.

We have, $(ab^3c^2)^3 = a^3b^9c^6$.

Then, by § 165, $\sqrt[3]{a^3b^9c^6} = ab^3c^2$.

2. Required the fifth root of $-32a^5$.

We have, $(-2a)^5 = -32a^5$.

Whence, $\sqrt[5]{-32a^5} = -2a$.

Similarly, to extract a root of any monomial:

Extract the required root of the absolute value of the numerical coefficient, and divide the exponent of each letter by the index of the required root.

Give to every even root of a positive term the sign $\pm$, and to every odd root of any term the sign of the term itself.

The sign $\pm$, called the *double sign*, is prefixed to an expression when we wish to indicate that it is either $+$ or $-$.

1. Find the square root of $9a^4b^6c^{10}$.

By the rule, $\sqrt{9a^4b^6c^{10}} = \pm 3a^2b^3c^5$.

It follows from §§ 167, 168, that, to find any root of a fraction, each of whose terms is a perfect power of the same degree as the index of the required root, *extract the required root of both numerator and denominator.*

2. $\sqrt[3]{-\dfrac{27\,a^3b^6}{64\,c^9}}=-\dfrac{\sqrt[3]{27\,a^3b^6}}{\sqrt[3]{64\,c^9}}=-\dfrac{3\,ab^2}{4\,c^3}.$

The root of a large number may sometimes be found by resolving it into its prime factors.

3. Find the square root of 254016.

We have, $\sqrt{254016}=\sqrt{2^6\times3^4\times7^2}=\pm2^3\times3^2\times7=\pm504.$

4. Find the value of $\sqrt[3]{72\times75\times135}.$

$$\begin{aligned}\sqrt[3]{72\times75\times135}&=\sqrt[3]{(2^3\times3^2)\times(3\times5^2)\times(3^3\times5)}\\&=\sqrt[3]{2^3\times3^6\times5^3}=2\times3^2\times5=90.\end{aligned}$$

EXERCISE 70

Find the values of the following:

1. $\sqrt{36\,x^2y^8}.$
2. $\sqrt[3]{64\,a^{12}b^3c^6}.$
3. $\sqrt[9]{-x^9y^{18}z^{27}}.$
4. $\sqrt{\dfrac{64\,x^2y^8}{49\,z^4}}.$
5. $\sqrt[3]{-\dfrac{27\,a^9}{125\,b^6}}.$
6. $\sqrt[4]{81\,n^{20}x^{12}y^4}.$
7. $\sqrt{121\,a^{12}b^{22}c^4}.$
8. $\sqrt[3]{-216\,x^{24}y^9z^{15}}.$
9. $\sqrt[7]{128\,m^{14}n^{21}}.$
10. $\sqrt[3]{-343\,x^{3m+9}y^{6n}}.$
11. $\sqrt[4]{625\,a^{16m}b^{4n}}.$
12. $\sqrt[4]{\dfrac{m^{20}}{256\,n^8}}.$
13. $\sqrt[6]{\dfrac{a^{12m}}{729\,b^{6n}}}.$
14. $\sqrt{2916}.$
15. $\sqrt{30625}.$
16. $\sqrt{86436}.$
17. $\sqrt{25\cdot36\cdot196}.$
18. $\sqrt[3]{27\cdot64\cdot8}.$
19. $\sqrt{250\cdot32\cdot45}.$
20. $\sqrt[3]{4\,ab\cdot144\,b^2c\cdot24\,a^2c^2}.$
21. $\sqrt[3]{252\,a^3\cdot245\,d^3\cdot150\,c^3}.$
22. $\sqrt[5]{59049}.$
23. $\sqrt[3]{112\cdot168\cdot252}.$
24. $\sqrt{(a^2-5\,a+6)(a^2+2\,a-8)(a^2+a-12)}.$
25. $\sqrt{(2\,a^2+7\,a-15)(8\,a^2+2\,a-21)(4\,a^2+27\,a+35)}.$

169. It may be proved that $\sqrt[n]{(a^n)^m}=\sqrt[n]{a^{mn}}=a^m=(\sqrt[n]{a^n})^m.$

Ex. Required the value of $\sqrt[5]{(32\,a^{10})^4}.$

We have, $\sqrt[5]{(32\,a^{10})^4}=(\sqrt[5]{32\,a^{10}})^4=(2\,a^2)^4=16\,a^8.$

This method of finding the root is shorter than raising $32a^{10}$ to the fourth power, and then taking the fifth root of the result.

EXERCISE 71

Find the values of the following:

1. $\sqrt[3]{(64a^3)^2}$.
2. $\sqrt{(4m^4)^7}$.
3. $\sqrt[4]{(16x^8y^{12})^5}$.
4. $\sqrt[5]{(-243a^5b^{25}c^{10})^3}$.
5. $\sqrt[6]{(64m^{12}n^6)^5}$.
6. $\sqrt[3]{\left(-\frac{8x^3}{27y^6}\right)^4}$
7. $\sqrt{(a^2-2ab+b^2)^3}$.

170. Square of a Polynomial. — We find by actual multiplication:

$$\begin{array}{llllll}
a & +b+c & & & & \\
a & +b+c & & & & \\
\hline
a^2+ & ab+ & ac & & & \\
 & +ab & & +b^2+ & bc & \\
 & & +ac & & +bc+c^2 & \\
\hline
a^2+2ab & +2ac & +b^2 & +2bc+c^2 & &
\end{array}$$

The result, for convenience of enunciation, may be written:

$$(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc.$$

In like manner we find:

$$(a+b+c+d)^2=a^2+b^2+c^2+d^2$$
$$+2ab+2ac+2ad+2bc+2bd+2cd;$$

and so on.

We then have the following rule:

The square of a polynomial is equal to the sum of the squares of its terms, together with twice the product of each term by each of the following terms.

Ex. Expand $(2x^2-3x-5)^2$.

The squares of the terms are $4x^4$, $9x^2$, and 25.

Twice the product of the first term by each of the following terms gives the results $-12x^3$ and $-20x^2$.

Twice the product of the second term by the following term gives the result $30x$.

Then, $(2x^2-3x-5)^2=4x^4+9x^2+25-12x^3-20x^2+30x$
$=4x^4-12x^3-11x^2+30x+25.$

EXERCISE 72

Square each of the following:

1. $a+b+c$.
2. $3a-x+2y$.
3. $r+2s-3t$.
4. $a+d+3d^2$.
5. $2c-5a+m$.
6. $3x^2-2x-1$.
7. $2x^2+x-5$.
8. $4m^2-4m+1$.
9. $a^2+2ab+b^2$.
10. a^3+a^2-2a-2.

171. Square Root of a Polynomial by Inspection.—In § 91, we showed how to find the square root of a trinomial perfect square.

The square roots of certain polynomials of the form

$$a^2+b^2+c^2+2ab+2ac+2bc$$

can be found by inspection.

Ex. Find the square root of

$$9x^2+y^2+4z^2+6xy-12xz-4yz.$$

We can write the expression as follows:

$$(3x)^2+y^2+(-2z)^2+2(3x)y+2(3x)(-2z)+2y(-2z).$$

By § 170, this is the square of $3x+y+(-2z)$.

Then, the square root of the expression is $3x+y-2z$.

(The result could also have been obtained in the form $2z-y-3x$.)

EXERCISE 73

Find the square roots of the following:

1. $a^2+b^2+c^2-2ab-2ac+2bc$.
2. $x^2+4y^2+9+4xy+6x+12y$.
3. $1+25m^2+36n^2-10m+12n-60mn$.
4. $a^2+81b^2+16+18ab-8a-72b$.
5. $9x^2+y^2+25z^2-6xy-30xz+10yz$.
6. $36m^2+64n^2+x^2+96mn-12mx-16nx$.
7. $16a^4+9b^4+81c^4+24a^2b^2+72a^2c^2+54b^2c^2$.
8. $25x^6+49y^{10}+36z^8-70x^3y^5+60x^3z^4-84y^5z^4$.

172. Square Root of a Polynomial Perfect Square, general method.

By § 170, $(a+b+c)^2 = a^2+2\,ab+b^2+2\,ac+2\,bc+c^2$

$$= a^2+(2\,a+b)b+(2\,a+2\,b+c)c. \qquad (1)$$

Then, if the square of a trinomial be arranged in order of powers of some letter:

I. The square root of the first term gives the first term of the root, a.

II. If from (1) we subtract a^2, we have

$$(2\,a+b)b+(2\,a+2\,b+c)c. \qquad (2)$$

The first term of this, when expanded, is $2\,ab$; if this be divided by twice the first term of the root, $2\,a$, we have the next term of the root, b.

III. If from (2) we subtract $(2\,a+b)b$, we have

$$(2\,a+2\,b+c)c. \qquad (3)$$

The first term of this, when expanded, is $2\,ac$; if this be divided by twice the first term of the root, $2\,a$, we have the last term of the root, c.

IV. If from (3) we subtract $(2\,a+2\,b+c)c$, there is no remainder.

Similar considerations hold with respect to the square of a polynomial of any number of terms.

173. The principles of § 172 may be used to find the square root of a polynomial perfect square of any number of terms. Let it be required to find the square root of

$$4\,x^4+12\,x^3-7\,x^2-24\,x+16.$$

$$\begin{array}{r|l}
 & 4\,x^4+12\,x^3-\ 7\,x^2-24\,x+16 \;\; \underline{|\; 2\,x^2+3\,x-4} \\
a^2=4\,x^4 & \\ \hline
2\,a+b=4\,x^2+3\,x & 12\,x^3-\ 7\,x^2-24\,x+16, \text{ 1st Rem.} \\
3\,x & 12\,x^3+\ 9\,x^2 \\ \hline
2\,a+2\,b+c=4\,x^2+6\,x-4 & -16\,x^2-24\,x+16, \text{ 2d Rem.} \\
-4 & -16\,x^2-24\,x+16 \\ \hline
\end{array}$$

The first term of the root is the square root of $4x^4$, or $2x^2$.

Subtracting the square of $2x^2$, $4x^4$, from the given expression, the first remainder is $12x^3-7x^2-24x+16$.

Dividing the first term of this by twice the first term of the root, $4x^2$, we have the next term of the root, $3x$ (§ 172, II).

Adding this to $4x^2$ gives $4x^2+3x$; multiplying the result by $3x$, and subtracting the product, $12x^3+9x^2$, from the first remainder, gives the second remainder, $-16x^2-24x+16$.

Dividing the first term of this by twice the first term of the root, $4x^2$, we have the last term of the root, -4 (§ 172, III).

If from the second remainder we subtract $(4x^2+6x-4)(-4)$, or $-16x^2-24x+16$, there is no remainder; then, $2x^2+3x-4$ is the required root (§ 172, IV).

The expressions $4x^2$ and $4x^2+6x$ are called *trial-divisors*, and $4x^2+3x$ and $4x^2+6x-4$ *complete divisors*.

We then have the following rule for extracting the square root of a polynomial perfect square:

Arrange the expression according to the powers of some letter.

Extract the square root of the first term, write the result as the first term of the root, and subtract its square from the given expression, arranging the remainder in the same order of powers as the given expression.

Divide the first term of the remainder by twice the first term of the root, and add the quotient to the part of the root already found, and also to the trial-divisor.

Multiply the complete divisor by the term of the root last obtained, and subtract the product from the remainder.

If other terms remain, proceed as before, doubling the part of the root already found for the next trial-divisor.

174. Examples.

1. Find the square root of $9x^4+30a^3x^2+25a^6$.

$$
\begin{array}{r|l}
 & 9x^4+30a^3x^2+25a^6 \;\; \underline{|\,3x^2+5a^3} \\
 & 9x^4 \\
\hline
6x^2+5a^3 & 30a^3x^2 \\
 & \underline{30a^3x^2+25a^6}
\end{array}
$$

It is usual, in practice, to omit those terms, after the first, in each remainder, which are merely repetitions of the terms in the given expres-

sion; thus, in the first remainder of Ex. 1, we leave out the term $25\,a^6$. It is also usual to leave out of the written work the multiplier of the complete divisor.

2. Find the square root of

$$20\,x^2-22\,x^3+1+28\,x^4+9\,x^6-8\,x-12\,x^5.$$

Arranging according to the descending powers of x, we have

$$\begin{array}{r|l}
 & 9\,x^6-12\,x^5+28\,x^4-22\,x^3+20\,x^2-8\,x+1 \;\big|\; \underline{3\,x^3-2\,x^2+4\,x-1} \\
 & 9\,x^6 \\ \hline
6\,x^3-2\,x^2 & -12\,x^5 \\
 & -12\,x^5+\;4\,x^4 \\ \hline
6\,x^3-4\,x^2+4\,x & 24\,x^4 \\
 & 24\,x^4-16\,x^3+16\,x^2 \\ \hline
6\,x^3-4\,x^2+8\,x-1 & -\;6\,x^3+\;4\,x^2 \\
 & -\;6\,x^3+\;4\,x^2-8\,x+1 \\ \hline
\end{array}$$

It will be observed that *each trial-divisor is equal to the preceding complete divisor with its last term doubled.*

If, in Ex. 2, we had written the expression

$$1-8\,x+20\,x^2-22\,x^3+28\,x^4-12\,x^5+9\,x^6,$$

the square root would have been obtained in the form $1-4\,x+2\,x^2-3\,x^3$, which is the negative of $3\,x^3-2\,x^2+4\,x-1$.

EXERCISE 74

Find the square roots of:

1. $x^4+4\,x^3+6\,x^2+4\,x+1.$
2. $4\,a^4-4\,a^3+17\,a^2-8\,a+16.$
3. $25\,x^4-30\,x^3-x^2+6\,x+1.$
4. $9\,x^4+24\,x^3+28\,x^2+16\,x+4.$
5. $36\,n^6+12\,n^4-60\,n^3+n^2-10\,n+25.$
6. $a^4-8\,a^3b+22\,a^2b^2-24\,ab^3+9\,b^4.$
7. $4\,x^4+12\,x^3y+13\,x^2y^2+6\,xy^3+y^4.$
8. $x^6+12\,x^5+36\,x^4-14\,x^3-84\,x^2+49.$
9. $16\,a^8-40\,a^6x^3+a^4x^6+30\,a^2x^9+9\,x^{12}.$
10. $x^6-2\,x^5-x^4+6\,x^3-3\,x^2-4\,x+4.$
11. $4\,a^6-20\,a^5+41\,a^4-52\,a^3+46\,a^2-24\,a+9.$

175. Square Root of Arithmetical Numbers. — The square root of 100 is 10; of 10000 is 100; etc.

Hence, the square root of a number between 1 and 100 is between 1 and 10; the square root of a number between 100 and 10000 is between 10 and 100; etc.

That is, the integral part of the square root of an integer of one or two digits, contains *one* digit; of an integer of three or four digits, contains *two* digits; and so on.

Hence, if a point be placed over every second digit of an integer, beginning at the units' place, the number of points shows the number of digits in the integral part of its square root.

176. Square Root of any Integral Perfect Square.

The square root of an integral perfect square may be found in the same way as the square root of a polynomial.

Required the square root of 106929.

$$\begin{array}{rr|l}
 & \dot{1}0\dot{6}9\dot{2}9 & 300+20+7 \\
 & a^2=90000 & =a+b+c \\
\hline
2a+b= & 600+20 \quad 16929 & \\
 & 20 \quad 12400 & \\
\hline
2a+2b+c= & 600+40+7 \quad 4529 & \\
 & 7 \quad 4529 & \\
\hline
\end{array}$$

Pointing the number in accordance with the rule of § 175, we find that there are three digits in its square root.

Let a represent the hundreds' digit of the root, with two ciphers annexed; b the tens' digit, with one cipher annexed; and c the units' digit.

Then, a must be the greatest multiple of 100 whose square is less than 106929; this we find to be 300.

Subtracting a^2, or 90000, from the given number, the result is 16929.

Dividing this remainder by $2a$, or 600, we have the quotient 28+; which suggests that b equals 20.

Adding this to $2a$, or 600, and multiplying the result by b, or 20, we have 12400; which, subtracted from 16929, leaves 4529.

Since this remainder equals $(2a+2b+c)c$ (§ 172, III), we can get c approximately by dividing it by $2a+2b$, or $600+40$.

Dividing 4529 by 640, we have the quotient $7+$; which suggests that c equals 7.

Adding this to $600+40$, multiplying the result by 7, and subtracting the product, 4529, there is no remainder.

Then, $300+20+7$, or 327, is the required square root.

177. Omitting the ciphers for the sake of brevity, and condensing the operation, we may arrange the work of the example of § 176 as follows:

$$\begin{array}{r|l}
 & \dot{1}0\dot{6}9\dot{2}9 \;\underline{|\;327} \\
 & 9 \\ \hline
62 & 169 \\
 & 124 \\ \hline
647 & 4529 \\
 & 4529 \\ \hline
\end{array}$$

The numbers 600 and 640 are called *trial-divisors*, and the numbers 620 and 647 are called *complete divisors*.

We then have the following rule for finding the square root of an integral perfect square:

Separate the number into periods by pointing every second digit, beginning with the units' place.

Find the greatest square in the left-hand period, and write its square root as the first digit of the root; subtract the square of the first root-digit from the left-hand period, and to the result annex the next period.

Divide this remainder, omitting the last digit, by twice the part of the root already found, and annex the quotient to the root, and also to the trial-divisor.

Multiply the complete divisor by the root-digit last obtained, and subtract the product from the remainder.

If other periods remain, proceed as before, doubling the part of the root already found for the next trial-divisor.

Note 1. It sometimes happens that, on multiplying a complete divisor by the digit of the root last obtained, the product is greater than the remainder. In such a case, the digit of the root last obtained is too great, and one less must be substituted for it.

Note 2. If any root-digit is 0, annex 0 to the trial-divisor, and annex to the remainder the next period. (See the illustrative example of § 179.)

178. *Ex.* Find the square root of 4624.

$$\begin{array}{r|l}
 & 4\dot{6}2\dot{4} \;\underline{|\,68} \\
 & 36 \\ \hline
128 & 1024 \\
 & 1024 \\ \hline
\end{array}$$

The greatest square in the left-hand period is 36.

Then the first digit of the root is 6.

Subtracting 6^2, or 36, from the left-hand period, the result is 10; to this we annex the next period, 24.

Dividing this remainder, omitting the last digit, or 102, by twice the part of the root already found, or 12, the quotient is 8; this we annex to the root, and also to the trial-divisor.

Multiplying the complete divisor, 128, by 8, and subtracting the product from the remainder, there is no remainder.

Then, 68 is the required square root.

179. **To find the square root of a number which is not integral.**

Ex. Find the square root of 49.449024.

We have, $\sqrt{49.449024}=\sqrt{\dfrac{49449024}{1000000}}=\dfrac{\sqrt{49449024}}{\sqrt{1000000}}.$

$$\begin{array}{r|l}
 & 4\dot{9}4\dot{4}9\dot{0}2\dot{4} \;\underline{|\,7032} \\
 & 49 \\ \hline
1403 & 4490 \\
 & 4209 \\ \hline
14062 & 28124 \\
 & 28124 \\ \hline
\end{array}$$

Since 14 is not contained in 4, we write 0 as the second root-digit, in the above example; we then annex 0 to the trial-divisor 14, and annex to the remainder the next period, 90. (See Note 2, § 177.)

Then, $\sqrt{49.449024}=\dfrac{7032}{1000}=7.032.$

The work may be arranged as follows:

$$\begin{array}{r|l}
 & 4\dot{9}.4\dot{4}9\dot{0}2\dot{4} \;\underline{|\,7.032} \\
 & 49 \\ \hline
1403 & 4490 \\
 & 4209 \\ \hline
14062 & 28124 \\
 & 28124 \\ \hline
\end{array}$$

Then, if a point be placed over every second digit of any number, beginning with the units' place, and extending in either direction, the rule of § 177 may be applied to the result and the decimal point inserted in its proper position in the root.

EXERCISE 75

Find the square roots of the following:

1. 5776.
2. 15376.
3. 67081.
4. 21904.
5. 508369.
6. 65.1249.
7. .156816.
8. .064516.
9. 3956.41.
10. 96.4324.
11. .00321489.

180. Approximate Square Roots.

If there is a final remainder, the number has no exact square root; but we may continue the operation by annexing periods of ciphers, and obtain an approximate root, correct to any desired number of decimal places.

Ex. Find the square root of 12 to four decimal places.

$$
\begin{array}{r|l}
 & 1\dot{2}.0\dot{0}0\dot{0}0\dot{0}0\dot{0} \;\underline{|\;3.4641+} \\
 & 9 \\
\hline
64 & 300 \\
 & 256 \\
\hline
686 & 4400 \\
 & 4116 \\
\hline
6924 & 28400 \\
 & 27696 \\
\hline
69281 & 70400
\end{array}
$$

181. The approximate square root of a fraction may be found by taking the square root of the numerator, and then of the denominator, and dividing the first result by the second.

If the denominator is not a perfect square, it is better to reduce the fraction to an equivalent fraction whose denominator is a perfect square.

Ex. Find the value of $\sqrt{\frac{3}{8}}$ to five decimal places.

We have, $\sqrt{\dfrac{3}{8}}=\sqrt{\dfrac{6}{16}}=\dfrac{\sqrt{6}}{\sqrt{16}}=\dfrac{2.44948+}{4}=.61237+.$

EXERCISE 78

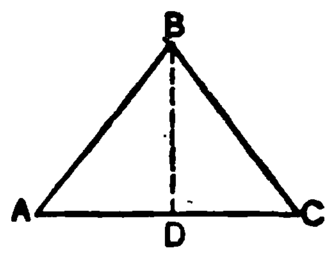
Find the first five figures of the square root of:

*1. 2.	5. .3.	9. 11.	13. 48.
2. 3.	6. $\frac{2}{5}$.	10. 12.	14. 50.
3. 5.	7. $\frac{8}{9}$.	11. .067.	15. $\frac{10}{27}$.
4. 7.	8. $\frac{7}{2}$.	12. $\frac{33}{25}$.	16. .056.

17. .00074.

18. The side of a square is 5; find the diagonal correct to four decimal places.

19. In an equilateral triangle, ABC, the altitude, DB, passes through the middle point of the base. If one side of the triangle is 8, find the altitude, correct to three decimal places.



182. Cube of a Binomial. — We find by actual multiplication:

$$\begin{aligned}
(a+b)^2 &= a^2 + 2\,ab + b^2 \\
&\quad\; a \;\; + b \\
\hline
&\quad\; a^3 + 2\,a^2b + ab^2 \\
&\qquad\qquad\;\; a^2b + 2\,ab^2 + b^3 \\
\hline
(a+b)^3 &= a^3 + 3\,a^2b + 3\,ab^2 + b^3
\end{aligned}$$

That is, the cube of the sum of two numbers is equal to the cube of the first, plus three times the square of the first times the second, plus three times the first times the square of the second, plus the cube of the second.

Again,
$$\begin{aligned}
(a-b)^2 &= a^2 - 2\,ab + b^2 \\
&\quad\; a \;\; - b \\
\hline
&\quad\; a^3 - 2\,a^2b + ab^2 \\
&\qquad\qquad - a^2b + 2\,ab^2 - b^3 \\
\hline
(a-b)^3 &= a^3 - 3\,a^2b + 3\,ab^2 - b^3
\end{aligned}$$

* The values of examples (1) and (2) are of frequent occurrence and are important.

That is, the cube of the difference of two numbers is equal to the cube of the first, minus three times the square of the first times the second, plus three times the first times the square of the second, minus the cube of the second.

1. Find the cube of $a+2b$.

We have, $(a+2b)^3=a^3+3a^2(2b)+3a(2b)^2+(2b)^3$
$$=a^3+6a^2b+12ab^2+8b^3.$$

2. Find the cube of $2x^3-5y^2$.

$$(2x^3-5y^2)^3=(2x^3)^3-3(2x^3)^2(5y^2)+3(2x^3)(5y^2)^2-(5y^2)^3$$
$$=8x^9-60x^6y^2+150x^3y^4-125y^6.$$

The cube of a *trinomial* may be found by the above method, if two of its terms be enclosed in parentheses; and regarded as a single term.

3. Find the cube of x^2-2x-1.

$$(x^2-2x-1)^3=[(x^2-2x)-1]^3$$
$$=(x^2-2x)^3-3(x^2-2x)^2+3(x^2-2x)-1$$
$$=x^6-6x^5+12x^4-8x^3-3(x^4-4x^3+4x^2)+3(x^2-2x)-1$$
$$=x^6-6x^5+12x^4-8x^3-3x^4+12x^3-12x^2+3x^2-6x-1$$
$$=x^6-6x^5+9x^4+4x^3-9x^2-6x-1.$$

EXERCISE 77

Cube each of the following:

1. a^2b-ab^2.
2. $x+4$.
3. $c-5$.
4. $3x^2+1$.
5. $\frac{x}{3}+\frac{3}{x}$.
6. $8x^2-3y^2$.
7. $2m-b$.
8. $9x^2-4y^2$.
9. $a+b+c$.
10. $a-2b-3c$.
11. $3d+4c^2+k$.
12. $3a^x+2b^y$.

183. Cube Root of a Polynomial. The cube roots of certain polynomials of the form

$$a^3+3a^2b+3ab^2+b^3$$

can be found by inspection.

Ex. Find the cube root of $8a^3-36a^2b^2+54ab^4-27b^6$.

We can write the expression as follows:

$$(2a)^3-3(2a)^2(3b^2)+3(2a)(3b^2)^2-(3b^2)^3.$$

By § 182, this is the cube of $2a-3b^2$.

Then, the cube root of the expression is $2a-3b^2$.

EXERCISE 78

Find the cube roots of the following:

1. $x^3+6x^2+12x+8.$
2. $27a^3-27a^2+9a-1.$
3. $m^6+15m^4+75m^2+125.$
4. $a^3-12a^2b+48ab^2-64b^3.$
5. $125x^3+150x^2y+60xy^2+8y^3.$
6. $216a^3-108a^2b+18ab^2-b^3.$
7. $27x^6-135x^5+225x^4-125x^3.$
8. $64t^3-144t^2u+108tu^2-27u^3.$
9. $8h^3+60h^2k+150hk^2+125k^3.$
10. $1-18x^2+108x^4-216x^6.$

XIII. THEORY OF EXPONENTS. IRRATIONAL NUMBERS

184. In the preceding portions of the work, an exponent has been considered only as a *positive integer*.

Thus, if m is a positive integer,

$$a^m = a\times a\times a\times\cdots \text{ to } m \text{ factors.} \qquad (\S\ 6)$$

The following results have been proved to hold for any positive integral values of m and n:

$$a^m\times a^n = a^{m+n}\ (\S\ 50). \qquad (1)$$

$$(a^m)^n = a^{mn}\ (\S\ 85). \qquad (2)$$

185. It is desirable to use exponents which are not positive integers; and we now proceed to assign to them the most convenient definitions and then prove the rules for their use. New meanings are conformed to the old laws. Thus our new exponents are to obey the old index law

$$a^m\times a^n = a^{m+n}. \qquad (1)$$

$$(a^m)^n = a^{mn}. \qquad (2)$$

Let it be required to find such a meaning for *fractional*, *negative* and *zero* exponents.

186. Meaning of a Fractional Exponent.

Let it be required to find a meaning for $a^{\frac{5}{3}}$.

If (1), § 184, is to hold for all values of m and n,

$$a^{\frac{5}{3}} \times a^{\frac{5}{3}} \times a^{\frac{5}{3}} = a^{\frac{5}{3}+\frac{5}{3}+\frac{5}{3}} = a^5.$$

Then, the *third power* of $a^{\frac{5}{3}}$ equals a^5.

Hence, $a^{\frac{5}{3}}$ may be defined as the *cube root* of a^5, or $a^{\frac{5}{3}} = \sqrt[3]{a^5}$.

Consider the general case:

Let it be required to find the meaning of $a^{\frac{p}{q}}$, where p and q are any positive integers.

If (1), § 184, is to hold for all values of m and n,

$$a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times \cdots \text{ to } q \text{ factors } = a^{\frac{p}{q}+\frac{p}{q}+\frac{p}{q}+\cdots \text{ to } q \text{ terms}} = a^{\frac{p}{q}\times q} = a^p.$$

Then, the qth power of $a^{\frac{p}{q}}$ equals a^p.

Hence, $a^{\frac{p}{q}}$ must be the qth root of a^p, or $a^{\frac{p}{q}} = \sqrt[q]{a^p}$.

Hence, in a fractional exponent, the numerator denotes a power, and the denominator a root.

For example, $a^{\frac{3}{4}} = \sqrt[4]{a^3}$; $b^{\frac{5}{2}} = \sqrt{b^5}$; $x^{\frac{1}{3}} = \sqrt[3]{x}$; etc.

This statement indicates that in expressions affected by a fractional exponent, both a root and a power are to be taken.

EXERCISE 79

Express the following with radical signs:

1. $a^{\frac{5}{8}}$. 3. $7m^{\frac{1}{4}}$. 5. $a^{\frac{2}{3}}b^{\frac{9}{8}}$. 7. $8a^{\frac{4}{5}}m^{\frac{3}{8}}$. 9. $x^{\frac{9}{4}}y^{\frac{1}{5}}z^{\frac{11}{6}}$.

2. $x^{\frac{7}{2}}$. 4. $5x^{\frac{8}{5}}$. 6. $x^{\frac{6}{7}}y^{\frac{10}{9}}$. 8. $10n^{\frac{5}{8}}x^{\frac{12}{7}}$. 10. $2a^{\frac{1}{m}}b^{\frac{n}{2}}c^{\frac{3p}{q}}$.

Express the following with fractional exponents:

11. $\sqrt[7]{x^6}$. 13. $\sqrt{m^3}$. 15. $3\sqrt[4]{b^9}$. 17. $9\sqrt[5]{m}\sqrt[8]{n^5}$.

12. $\sqrt[6]{a}$. 14. $\sqrt[3]{n^4}$. 16. $4\sqrt[9]{y^2}$. 18. $\sqrt[6]{x^7}\sqrt[9]{y^8}$.

19. $\sqrt[8]{a}\sqrt[7]{b^{10}}$. 20. $\sqrt[3]{x^m}\sqrt{y^{3n}}\sqrt[2p]{z^q}$.

187. Meaning of a Zero Exponent.

If (1), § 184, is to hold for all values of m and n, we have
$$a^m \times a^0 = a^{m+0} = a^m.$$

Whence, $$a^0 = \frac{a^m}{a^m} = 1.$$

This meaning may be illustrated as follows:

Arithmetically, $$\frac{a^3}{a^3} = 1,$$

Algebraically, $$a^3 \div a^3 = a^0.$$

Therefore, $$a^0 = 1. \qquad (\S 4, \text{ax. } 4)$$

We must then define a^0 as being equal to 1.

188. Meaning of a Negative Exponent.

Let it be required to find a meaning for a^{-3}.

If (1), § 184, is to hold for all values of m and n,
$$a^{-3} \times a^3 = a^{-3+3} = a^0 = 1 \ (\S\ 187).$$

Whence, $$a^{-3} = \frac{1}{a^3}.$$

Consider the general case:

Let it be required to find the meaning of a^{-s}, where s represents a positive integer or a positive fraction.

If (1), § 184, is to hold for all values of m and n,
$$a^{-s} \times a^s = a^{-s+s} = a^0 = 1 \ (\S\ 187).$$

Whence, $$a^{-s} = \frac{1}{a^s}.$$

We must then define a^{-s} as being equal to 1 divided by a^s.

For example, $a^{-2} = \frac{1}{a^2}$; $a^{-\frac{2}{3}} = \frac{1}{a^{\frac{2}{3}}}$; $3\,x^{-1}y^{-\frac{1}{2}} = \frac{3}{xy^{\frac{1}{2}}}$; etc.

189. It follows from § 188 that

Any *factor* of the numerator of a fraction may be transferred to the denominator, or any *factor* of the denominator to the numerator, if the sign of its exponent be changed.

Thus, $\dfrac{a^2b^3}{cd^4} = \dfrac{b^3}{a^{-2}cd^4} = \dfrac{a^2b^3c^{-1}}{d^4} = \dfrac{a^2d^{-4}}{b^{-3}c}$, etc.

EXERCISE 80

Express with positive exponents:

1. $x^{-4}y^3$.
2. $a^{\frac{5}{2}}b^{-8}$.
3. $m^{-\frac{1}{4}}n^{\frac{1}{6}}$.
4. $3\,n^{-1}x$.
5. $a^{-2}m^{-3}$.
6. $m^{-\frac{3}{10}}x^9$.
7. $4\,a^{-\frac{4}{3}}n^{-5}$.
8. $5\,x^4y^{-\frac{8}{9}}z^{-7}$.
9. $m^{-\frac{2}{7}}n^{-9}$.
10. $8\,a^{-\frac{5}{4}}b^{-10}c^7$.
11. $6\,m^{-\frac{9}{8}}n^{-\frac{7}{6}}x^{\frac{3}{4}}$.
12. $7\,a^{-\frac{5}{2}}n^{-6}x^{-\frac{6}{5}}$.

Transfer all *literal* factors from the denominators to the numerators in the following:

13. $\dfrac{1}{x^8}$.
14. $\dfrac{a^{\frac{4}{5}}}{b^{-\frac{7}{3}}}$.
15. $\dfrac{1}{8\,m^{-2}n^{\frac{3}{2}}}$.
16. $\dfrac{3}{ax^{-4}}$.
17. $\dfrac{z^{-\frac{2}{5}}}{x^{-\frac{2}{7}}y^3}$.
18. $\dfrac{2\,m^5}{5\,np^{-1}}$.
19. $\dfrac{7\,a^{\frac{1}{3}}b^{-8}}{6\,c^{\frac{1}{6}}d^{\frac{8}{9}}}$.
20. $\dfrac{9m^{-7}x^{-\frac{5}{4}}}{4\,n^{-\frac{5}{8}}y^{-9}}$.

Transfer all literal factors from the numerators to the denominators in the following:

21. $\dfrac{4\,a^7}{3}$.
22. $\dfrac{x^{\frac{4}{5}}}{y^{\frac{3}{7}}}$.
23. $\dfrac{m^{-2}n^{-\frac{1}{6}}}{2}$.
24. $\dfrac{xy^{-5}}{5}$.
25. $\dfrac{6\,a^{-\frac{2}{3}}b^3}{c^6}$.
26. $\dfrac{a^8m^{\frac{3}{7}}}{n^{-7}}$.
27. $\dfrac{9\,m^{-6}n^{-\frac{7}{8}}}{x^9y^{-\frac{1}{3}}}$.
28. $\dfrac{8\,a^{-1}b^{\frac{8}{9}}}{5\,c^{-8}d^{-\frac{7}{8}}}$.

190. Since this is an elementary course, the student is only expected to *read* §§ 191 to 194, then use § 196 in applying the principles involved.

191. We obtained the definitions of fractional, zero, and negative exponents by supposing equation (1), § 184, to hold for such exponents.

Then, for any values of m and n,

$$a^m \times a^n = a^{m+n}. \qquad (1)$$

The formal proof of this result for positive or negative, integral or fractional, values of m and n will be found in the Second Course.

192. $\dfrac{a^m}{a^n}=a^{m-n}$ *for all values of m and n.*

By § 189, $\dfrac{a^m}{a^n}=a^m\times a^{-n}=a^{m-n}$, by (1), § 191.

The proof of this result in the case where m and n are positive integers, and $m>n$, is given in § 63.

193. To prove equation (2), § 184, for any values of m and n, considering three cases, in each of which m may have any value, positive or negative, integral or fractional.

I. Let n be a positive integer.

The proof given in § 85 holds if n is a positive integer, whatever the value of m.

II. Let $n=\dfrac{p}{q}$, where p and q are positive integers.

Then, by the definition of § 186,

$$(a^m)^{\frac{p}{q}}=\sqrt[q]{(a^m)^p}=\sqrt[q]{a^{mp}}\ (\S\ 193,\ \text{I})=a^{\frac{mp}{q}}.$$

III. Let $n=-s$, where s is a positive number.

Then, by the definition of § 188,

$$(a^m)^{-s}=\frac{1}{(a^m)^s}=\frac{1}{a^{ms}}\ (\S\ 193,\ \text{I or II})=a^{-ms}.$$

Therefore, the result holds for all values of m and n.

194. To prove the result

$$(ab)^n=a^nb^n,$$

for any fractional or negative value of n.

The proof of this result in the case where n is any positive integer, was given in § 86.

I. Let $n=\dfrac{p}{q}$, where p and q are any positive integers.

By § 193, $\qquad [(ab)^{\frac{p}{q}}]^q=(ab)^p=a^pb^p\ (\S\ 86). \qquad (1)$

By § 86, $\qquad (a^{\frac{p}{q}}b^{\frac{p}{q}})^q=(a^{\frac{p}{q}})^q(b^{\frac{p}{q}})^q=a^pb^p. \qquad (2)$

From (1) and (2), $[(ab)^{\frac{p}{q}}]^q=(a^{\frac{p}{q}}b^{\frac{p}{q}})^q.$

Taking the qth root of both members, we have

$$(ab)^{\frac{p}{q}}=a^{\frac{p}{q}}b^{\frac{p}{q}}.$$

II. Let $n=-s$, where s is any positive integer or positive fraction.

Then, $(ab)^{-s}=\dfrac{1}{(ab)^s}=\dfrac{1}{a^sb^s}$(§ 86, or § 194, I)$=a^{-s}b^{-s}$.

195. The value of a numerical expression affected with a fractional exponent may be found by first, if possible, extracting the root indicated by the denominator, and then raising the result to the power indicated by the numerator.

Ex. Find the value of $(-8)^{\frac{2}{3}}$.

By § 193, $(-8)^{\frac{2}{3}}=[(-8)^{\frac{1}{3}}]^2=(\sqrt[3]{-8})^2=(-2)^2=4.$

This holds only for *real* numbers.

196. Remember that the exponent laws given in §§ 50, 63, 88, 168, hold whether the exponents be integral or fractional either positive or negative, i.e. In multiplication, add exponents of like letters; in division, subtract exponents of like letters in the divisor from those in the dividend; in involution, multiply the exponents by the index of the power; in evolution, divide the exponents by the index of the root.

1. Find the value of $a^2\times a^{-5}$.

We have, $\qquad a^2\times a^{-5}=a^{2-5}=a^{-3}.$

2. Find the value of $a\times\sqrt{a^5}$.

By § 186, $\qquad a\times\sqrt{a^5}=a\times a^{\frac{5}{2}}=a^{1+\frac{5}{2}}=a^{\frac{7}{2}}.$

3. Multiply $a+2a^{\frac{2}{3}}-3a^{\frac{1}{3}}$ by $2-4a^{-\frac{1}{3}}-6a^{-\frac{2}{3}}$.

$$\begin{array}{llll}
a+2a^{\frac{2}{3}} & -3a^{\frac{1}{3}} & & \\
2-4a^{-\frac{1}{3}} & -6a^{-\frac{2}{3}} & & \\
\hline
2a+4a^{\frac{2}{3}} & -6a^{\frac{1}{3}} & & \\
\quad -4a^{\frac{2}{3}} & -8a^{\frac{1}{3}} & +12 & \\
 & -6a^{\frac{1}{3}} & -12 & +18a^{-\frac{1}{3}} \\
\hline
2a & -20a^{\frac{1}{3}} & & +18a^{-\frac{1}{3}}
\end{array}$$

It must be carefully observed, in examples like the foregoing, that the zero power of any number equals 1 (§ 187).

4. Find the value of $\dfrac{a^{-3}}{\sqrt[5]{a^2}}$.

$$\frac{a^{-3}}{\sqrt[5]{a^2}}=\frac{a^{-3}}{a^{\frac{2}{5}}}=a^{-3-\frac{2}{5}}=a^{-\frac{17}{5}}.$$

5. Divide $18\,xy^{-2}-23+x^{-\frac{1}{2}}y+6\,x^{-1}y^2$ by $3\,x^{\frac{3}{4}}y^{-1}+x^{\frac{1}{4}}-2\,x^{-\frac{1}{4}}y$.

$$\begin{array}{r|l}
18\,xy^{-2}-23+x^{-\frac{1}{2}}y+6\,x^{-1}y^2 & 3\,x^{\frac{3}{4}}y^{-1}+x^{\frac{1}{4}}-2\,x^{-\frac{1}{4}}y \\ \cline{2-2}
\underline{18\,xy^{-2}+6\,x^{\frac{1}{2}}y^{-1}-12\qquad\qquad\quad} & 6\,x^{\frac{1}{4}}y^{-1}-2\,x^{-\frac{1}{4}}-3\,x^{-\frac{3}{4}}y \\
-6\,x^{\frac{1}{2}}y^{-1}-11+x^{-\frac{1}{2}}y+6\,x^{-1}y^2 & \\
\underline{-6\,x^{\frac{1}{2}}y^{-1}-2+4\,x^{-\frac{1}{2}}y\qquad\qquad} & \\
-9-3\,x^{-\frac{1}{2}}y+6\,x^{-1}y^2 & \\
\underline{-9-3\,x^{-\frac{1}{2}}y+6\,x^{-1}y^2} &
\end{array}$$

It is important to arrange the dividend, divisor, and each remainder in the same order of powers of some common letter.

6. Find the value of $(a^2)^{-5}$.

We have, § 193, III, $(a^2)^{-5}=a^{2\times-5}=a^{-10}$.

7. Find the value of $(a^{-3})^{-\frac{1}{3}}$.

$$(a^{-3})^{-\frac{1}{3}}=a^{-3\times-\frac{1}{3}}=a.$$

8. Find the value of $(\sqrt{a})^{\frac{2}{3}}$.

$$(\sqrt{a})^{\frac{2}{3}}=(a^{\frac{1}{2}})^{\frac{2}{3}}=a^{\frac{1}{2}\times\frac{2}{3}}=a^{\frac{1}{3}}.$$

EXERCISE 81

Multiply the following:

1. a^4 by $a^{\frac{1}{2}}$.
2. a^6 by a^{-4}.
3. a^{-5} by a^3.
4. $2\,a^5$ by a^{-5}.
5. $4\,a^{\frac{1}{3}}$ by $6\,a^{\frac{2}{3}}$.
6. $12\,x^{-1}$ by $\sqrt{x}$.
7. $a^{-5}\sqrt[4]{x^3}$ by $a^{-8}\sqrt[3]{x^4}$.
8. $m^{-1}k^{-\frac{3}{10}}$ by $\dfrac{1}{3\,m^{-\frac{9}{7}}}k^{\frac{1}{6}}$.
9. $\sqrt[6]{a}$ by $a^{\frac{3}{4}}$.
10. $3\,x^{-\frac{1}{4}}y$ by $4\,x^{\frac{1}{2}}y^{-1}$.

Divide the following:

11. x^4 by x^6.
12. $a^{\frac{5}{8}}$ by $a^{\frac{1}{2}}$.
13. c by $c^{-\frac{4}{3}}$.
14. $4\,x^{-4}$ by $7\,x^{-6}$.
15. $6\sqrt{x^3}$ by $3\sqrt{x}$.
16. $3\,x^{-\frac{2}{3}}y^{-1}$ by $4\,x^{-\frac{1}{6}}y$.
17. $12\,a^{-2}b^4c$ by $6\,a^{-2}d$.
18. $14\,a^{-1}b^{\frac{1}{3}}$ by $-7\,ab^{\frac{1}{3}}$.

Simplify:

19. $(a^{\frac{1}{2}}+a^{\frac{1}{4}}+1)a^{-\frac{1}{2}}$.
20. $(x^{-2}+2\,x^{-1}+1)4\,x^2$.
21. $(4\,a^{-4}+10\,a^{-2}+25)(2\,a^{-2}-5)$.
22. $\dfrac{x^{-4}+2\,a^{-1}x^{-2}-15\,a^{-2}}{x^{-2}+5a^{-1}}$.
23. $(x^2-1)\div(x^{\frac{1}{2}}+1)$.
24. $n^{-6}-n^{-3}\div n^{-3}-1$.
25. $(n^{-6}-n^{-3})\div(n^{-3}-1)$.
26. $(2\,n^{-\frac{3}{4}}-5-6\,n^{\frac{3}{4}})(3\,n^{-\frac{3}{4}}-4)$.
27. $(a^{-\frac{4}{3}}+2\,a^{-\frac{2}{3}}b^{-2}+9\,b^{-4})\div(a^{-1}+2\,a^{-\frac{2}{3}}b^{-1}+3\,a^{-\frac{1}{3}}b^{-2})$.

Find by inspection:

28. $(a^{\frac{1}{2}}+b^{\frac{1}{2}})(a^{\frac{1}{2}}-b^{\frac{1}{2}})$. (§ 89.)
29. $(a^{\frac{1}{2}}+b^{\frac{1}{2}})^2$. (§ 91.)
30. $(a^{-1}+2\,b^{-2})(a^{-1}-6\,b^{-2})$.
31. $(2\,a^{\frac{1}{3}}+3\,b^{\frac{4}{3}})(5\,a^{\frac{1}{3}}+3\,b^{\frac{4}{3}})$.
32. $(a+8\,b)\div(a^{\frac{1}{3}}+2\,b^{\frac{1}{3}})$.
33. $(16\,a-25\,c)\div(4\,a^{\frac{1}{2}}-5\,c^{\frac{1}{2}})$.

34. Factor $4\,a-4\,a^{\frac{1}{2}}+1$. (Call the first term a perfect square.)

35. $49-36\,b$. Factor, using § 89.

36. $(c^{\frac{1}{3}}-2\,d^{\frac{1}{3}})^3=$?

Supply the missing term in the following trinomial squares:

37. $a+2\,a^{\frac{1}{2}}b^{\frac{1}{2}}$.
38. $9\,c+12\,c^{\frac{1}{2}}$.
39. $x+9$.
40. $25\,a^{\frac{1}{2}}-10\,a^{\frac{1}{4}}$.
41. $16\,c^{\frac{1}{2}}+36\,d^{\frac{1}{2}}$.

Find the values of:

42. $(x^{-\frac{2}{3}}y^{\frac{6}{5}})^{\frac{1}{4}}$.
43. $(a^{\frac{1}{4}}x^{\frac{3}{2}})^{-\frac{2}{3}}$.
44. $(n^{-3}\sqrt{x^5})^{-4}$.
45. $(\sqrt[4]{a^3}\sqrt[5]{b^{-8}})^{\frac{5}{6}}$.
46. $81^{-\frac{1}{2}}$.
47. $(-32)^{\frac{4}{5}}$.
48. $36^{-\frac{3}{2}}$.
49. $(-8)^{\frac{10}{3}}$.
50. $729^{-\frac{5}{6}}$.
51. $512^{\frac{5}{9}}$.

Extract the square root of:

52. $16\,a^{-6}m^{\frac{1}{4}}$.

53. $4\,a^{-\frac{8}{5}}+20\,a^{-\frac{6}{5}}+21\,a^{-\frac{4}{5}}-10\,a^{-\frac{2}{5}}+1$.

54. $4\,a^{\frac{1}{2}}+4\,a^{\frac{1}{4}}-19-10\,a^{-\frac{1}{4}}+25\,a^{-\frac{1}{2}}$.

Simplify the following, expressing all the results with positive exponents:

55. $\dfrac{a^{-4}b^2}{c^{-1}}\cdot\dfrac{c^2}{ab}$.

56. $\dfrac{a^n\cdot(a^{n-1})^n}{a^{n+1}\cdot a^{n-1}}$.

57. $\dfrac{a^{\frac{9}{10}}}{b^{\frac{5}{6}}c^{\frac{1}{9}}}\cdot\dfrac{b^{\frac{3}{4}}c^{\frac{7}{12}}}{a^{\frac{8}{15}}}$.

58. $(2^{n+4}-2\cdot 2^n)(2^{-2}\cdot 2^{-n-2})$.

59. $\dfrac{a+b}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}-\dfrac{a^{\frac{3}{2}}+b^{\frac{3}{2}}}{a-b}$.

60. $\dfrac{a^{\frac{1}{2}}+b^{\frac{1}{2}}}{a^{\frac{1}{2}}-b^{\frac{1}{2}}}-\dfrac{a^{-\frac{1}{2}}+b^{-\frac{1}{2}}}{a^{-\frac{1}{2}}-b^{-\frac{1}{2}}}$.

61. $(a^{\frac{1}{2}}+a^{-\frac{1}{2}})^2-4=?$

62. $\dfrac{\left(\dfrac{e^x+e^{-x}}{2}\right)^2-1}{\left(\dfrac{e^x-e^{-x}}{2}\right)^2}=?$

Extract the square root of:

63. $a+2\,a^{\frac{1}{2}}b^{\frac{1}{2}}+b$.

64. $2+2\sqrt{6}+3$. (See § 91.)

65. Is $6+2(18)^{\frac{1}{2}}+3$ a perfect square?

IRRATIONAL NUMBERS

197. A **Surd** is the indicated root of a number, or expression, which is not a perfect power of the degree denoted by the index of the radical sign; as $\sqrt{2}$, $\sqrt[3]{5}$, $\sqrt[4]{x+y}$, or $(3)^{\frac{1}{2}}$.

198. A monomial is said to be *rational* when it is rational and integral (§ 57), or else a fraction whose terms are rational and integral.

A polynomial is said to be rational when each of its terms is rational.

An expression is said to be *irrational* when it involves surds; as $2+\sqrt{3}$, or $\sqrt{a+1}-\sqrt{a}$.

199. A *rational number* is a positive or negative integer, or a positive or negative fraction.

A numerical expression involving surds is an *irrational number.**

200. If a surd is in the form $b\sqrt[n]{a}$, b is called the *coefficient* of the surd, and n the *index*.

201. The *degree* of a surd is denoted by its index; thus, $\sqrt[3]{5}$ is a surd of the third degree.

A *quadratic surd* is a surd of the second degree.

For example, the square roots of positive arithmetical numbers *not* belonging to the set

1, 4, 9, 16, 25, 36, 49, etc., are *quadratic surds.*

The cube roots of arithmetical numbers *not* belonging to the set

1, 8, 27, 64, 125, etc., are *cubic surds.*

REDUCTION OF A SURD TO ITS SIMPLEST FORM

202. A surd is said to be in its *simplest form* when the expression under the radical sign is rational and integral (§ 57), is not a perfect power of the degree denoted by any factor of the index of the surd, and has no factor which is a perfect power of the same degree as the surd.

203. CASE I. *When the expression under the radical sign is a perfect power of the degree denoted by a factor of the index.*

Ex. 1. Reduce $\sqrt[6]{8}$ to its simplest form.

We have, $\sqrt[6]{8}=\sqrt[6]{2^3}=2^{\frac{3}{6}}(§\ 186)=2^{\frac{1}{2}}=\sqrt{2}.$

Ex. 2. Reduce $\sqrt[6]{16}$ to its simplest form.

We have, $\sqrt[6]{16}=[(2^4)^{\frac{1}{2}}]^{\frac{1}{3}}=(2^2)^{\frac{1}{3}}=4^{\frac{1}{3}}=\sqrt[3]{4}.$

In many radical forms, operations are more simple when the quantities are reduced to forms with fractional exponents.

* Note that we do not define irrational number. The two most important irrationals, — π and ϵ (the base of a system of logarithms), — have been proved not to involve surds.

EXERCISE 82

Reduce the following to their simplest forms:

1. $\sqrt[4]{25}$.
2. $\sqrt[10]{4}$.
3. $\sqrt[8]{121}$.
4. $\sqrt[9]{125}$.
5. $\sqrt[16]{49}$.
6. $\sqrt[16]{81}$.
7. $\sqrt[14]{64}$.
8. $\sqrt[10]{81}$.
9. $\sqrt[10]{243}$.
10. $\sqrt[15]{343}$.
11. $\sqrt[4]{144\,x^6y^8}$.
12. $\sqrt[6]{27\,n^6x^{12}}$.
13. $\sqrt[12]{216\,a^9x^3}$.
14. $\sqrt[12]{64\,a^6b^{18}}$.
15. $\sqrt[18]{8\,a^{18}b^{15}}$.
16. $\sqrt[12]{625\,x^{12}y^8}$.

204. CASE II. *When the expression under the radical sign is rational and integral, and has a factor which is a perfect power of the same degree as the surd.*

1. Reduce $\sqrt[3]{54}$ to its simplest form.

We have, $\sqrt[3]{54}=(27\cdot 2)^{\frac{1}{3}}=27^{\frac{1}{3}}\cdot 2^{\frac{1}{3}}=(3^3)^{\frac{1}{3}}\cdot 2^{\frac{1}{3}}=3\cdot 2^{\frac{1}{3}}=3\sqrt[3]{2}$.

2. Reduce $\sqrt{3\,a^3b-12\,a^2b^2+12\,ab^3}$ to its simplest form.

$$\sqrt{3\,a^3b-12\,a^2b^2+12\,ab^3}=\sqrt{(a^2-4\,ab+4\,b^2)3\,ab}$$
$$=\sqrt{a^2-4\,ab+4\,b^2}\sqrt{3\,ab}=(a-2\,b)\sqrt{3\,ab}.$$

We then have the following rule:

Resolve the expression under the radical sign into two factors, the second of which contains no factor which is a perfect power of the same degree as the surd.

Extract the required root of the first factor, and multiply the result by the indicated root of the second.

If the expression under the radical sign has a numerical factor which cannot be readily factored by inspection, it is convenient to resolve it into its prime factors.

3. Reduce $\sqrt[3]{1944}$ to its simplest form.

$\sqrt[3]{1944}=\sqrt[3]{2^3\times 3^5}=(2^3\cdot 3^5)^{\frac{1}{3}}=(2^3\cdot 3^3)^{\frac{1}{3}}\cdot(3^2)^{\frac{1}{3}}=2\cdot 3\cdot(3^2)^{\frac{1}{3}}=6\sqrt[3]{9}$.

4. Reduce $\sqrt{125\times 147}$ to its simplest form.

$\sqrt{125\times 147}=\sqrt{5^3\times 3\times 7^2}=\sqrt{5^2\times 7^2}\times\sqrt{5\times 3}=5\times 7\times\sqrt{15}=35\sqrt{15}$.

EXERCISE 83

Reduce the following to their simplest forms:

1. $(45)^{\frac{1}{2}}$.
4. $\sqrt{75}$.
7. $\sqrt[3]{54}$.
10. $\sqrt[5]{64}$.
2. $(12)^{\frac{1}{2}}$.
5. $\sqrt{15}$.
8. $(128)^{\frac{1}{3}}$.
11. $\sqrt{500\,a^3b^6}$.
3. $(96)^{\frac{1}{2}}$.
6. $(27)^{\frac{1}{3}}$.
9. $\sqrt[6]{192}$.
12. $\sqrt{98\,x^6y-49\,x^4y^2}$.
13. $[(a+3)(a^2-9)]^{\frac{1}{2}}$.
14. $\sqrt{(2\,a^2+2\,ay-4\,y^2)(3\,a-3\,y)}$.
15. $[(x^2+9)x]^{\frac{1}{2}}$.
19. $\sqrt[3]{63\,x^2y\cdot 75\,x^2y^3\cdot 98\,z}$.
16. $\sqrt{(x^2+6\,x+9)x}$.
20. $\sqrt{98\cdot 196}$.
17. $\sqrt{(a^2+2\,ab+4\,b^2)a^2}$.
21. $(5145)^{\frac{1}{2}}$.
18. $\sqrt{a^2+4\,ab+4\,b^2(a-b)}$.
22. $\sqrt{3\,a^3-24\,a^2+48\,a}$.
23. $\sqrt{18\,a^3b+60\,a^2b^2+50\,ab^3}$.
24. $\sqrt{(6\,x^2+5\,xy-4\,y^2)(3\,x^2-2\,xy-8\,y^2)}$.

205. Case III. *When the expression under the radical sign is a fraction.*

In this case, we multiply both terms of the fraction by such an expression as will make the denominator a perfect power of the same degree as the surd, and then proceed as in § 204.

Ex. Reduce $\sqrt{\dfrac{9}{8\,a^3}}$ to its simplest form.

Multiplying both terms of the fraction by $2\,a$, we have

$$\sqrt{\frac{9}{8\,a^3}}=\sqrt{\frac{9\times 2\,a}{16\,a^4}}=\sqrt{\frac{9}{16\,a^4}\times 2\,a}=\sqrt{\frac{9}{16\,a^4}}\times\sqrt{2\,a}=\frac{3}{4\,a^2}\sqrt{2\,a}.$$

EXERCISE 84

Reduce the following to their simplest forms:

1. $\sqrt{\frac{5}{2}}$.
3. $\sqrt{\frac{25}{8}}$.
5. $\sqrt{\frac{23}{27}}$.
7. $\sqrt[3]{\frac{4}{9}}$.
9. $\sqrt[5]{\frac{2}{3}}$.
2. $\sqrt{\frac{9}{5}}$.
4. $\sqrt{\frac{17}{18}}$.
6. $\sqrt{\frac{11}{12}}$.
8. $\sqrt[4]{\frac{3}{8}}$.
10. $\sqrt{\frac{5}{32}}$.

11. $\sqrt{\frac{5}{16}}$. 12. $\sqrt{\dfrac{2a-b}{2a+b}}$. 13. $\sqrt{\dfrac{a^2-ab+b^2}{a+b}}$.

14. $\dfrac{1}{a-3}\sqrt{\dfrac{2a^2-a-15}{a-3}}$. 15. $\dfrac{x^3}{x^2-5x+6}\sqrt{\dfrac{3x^2-18x+27}{x^3}}$.

206. To Introduce the Coefficient of a Surd under the Radical Sign.

The coefficient of a surd may be introduced under the radical sign by raising it to the power denoted by the index.

Ex. Introduce the coefficient of $2\sqrt[3]{3}$ under the radical sign.

$$2\sqrt[3]{3}=\sqrt[3]{8}\times\sqrt[3]{3}=\sqrt[3]{8\times 3}=\sqrt[3]{24}.$$

A rational expression (§ 198) may be expressed in the form of a surd of any degree by raising it to the power denoted by the index, and writing the result under the corresponding radical sign.

EXERCISE 85

Introduce the coefficients of the following under the radical signs:

1. $2\sqrt{3}$. 3. $6\sqrt{6}$. 5. $2\sqrt[3]{5}$. 7. $(2a+b)\sqrt{\dfrac{1}{4a^2-b^2}}$.

2. $5\sqrt{2}$. 4. $5\sqrt[3]{3}$. 6. $3\sqrt{\frac{1}{3}}$.

8. $(3x-2y)\sqrt{x}$. 10. $\dfrac{a-3}{a+3}\sqrt{\dfrac{a^2+a-6}{a^2-a-6}}$.

9. $(a+b)\sqrt{\dfrac{a-b}{a+b}}$. 11. $\dfrac{1}{2a-3b}\sqrt{4a^2-9b^2}$.

ADDITION AND SUBTRACTION OF SURDS

207. Similar Surds are surds which do not differ at all, or differ only in their coefficients; as $2\sqrt[3]{ax^2}$ and $3\sqrt[3]{ax^2}$.

Dissimilar Surds are surds which are not similar.

208. To add or subtract *similar surds* (§ 207), add or subtract their coefficients, and multiply the result by their common surd part.

1. Required the sum of $\sqrt{20}$ and $\sqrt{45}$.

Reducing each surd to its simplest form (§ 204),

$$\sqrt{20}+\sqrt{45}=\sqrt{4\times 5}+\sqrt{9\times 5}=2\sqrt{5}+3\sqrt{5}=5\sqrt{5}.$$

2. Simplify $\sqrt{\frac{1}{2}}+\sqrt{\frac{2}{3}}-\sqrt{\frac{9}{8}}$.

$$\sqrt{\frac{1}{2}}+\sqrt{\frac{2}{3}}-\sqrt{\frac{9}{8}}=\sqrt{\frac{1}{4}\times 2}+\sqrt{\frac{1}{9}\times 6}-\sqrt{\frac{9}{16}\times 2}$$
$$=\frac{1}{2}\sqrt{2}+\frac{1}{3}\sqrt{6}-\frac{3}{4}\sqrt{2}=\frac{1}{3}\sqrt{6}-\frac{1}{4}\sqrt{2}.$$

We then have the following rule:

Reduce each surd to its simplest form.

Add or subtract the similar surds, and indicate the addition or subtraction of the dissimilar.

EXERCISE 86

Simplify the following:

1. $\sqrt{12}+\sqrt{48}$.
2. $\sqrt{50}-\sqrt{18}$.
3. $2\sqrt{2}+\sqrt{128}-\sqrt{98}$.
4. $\sqrt{125}+\sqrt{180}+\sqrt{80}$.
5. $\sqrt[3]{2}+\sqrt[3]{16}$.
6. $\sqrt[3]{40}+\sqrt[3]{135}$.
7. $\sqrt[4]{32}-\sqrt[4]{162}$.
8. $\sqrt{250}+\sqrt{490}-\sqrt{10}$.
9. $\sqrt{44}-\sqrt{275}+\sqrt{891}$.
10. $\sqrt{32}+\sqrt{48}+\sqrt{80}$.
11. $\sqrt{5}+\sqrt{245}-\sqrt{320}$.
12. $\sqrt{\frac{8}{5}}+\sqrt{\frac{9}{10}}-\sqrt{\frac{5}{8}}$.
13. $\sqrt{\frac{16}{15}}-\sqrt{\frac{20}{3}}+\sqrt{\frac{3}{5}}$.
14. $\sqrt[3]{\frac{1}{4}}+\sqrt[3]{\frac{1}{32}}+\sqrt[3]{\frac{2}{3}}$.
15. $\frac{\sqrt{175}}{5}-\frac{\sqrt{112}}{3}-\frac{\sqrt{44}}{2}$.
16. $7\sqrt{27}-\sqrt{75}-24\sqrt{\frac{1}{12}}-27\sqrt{\frac{1}{27}}$.
17. $\sqrt[3]{\frac{2}{9}}+\sqrt[3]{\frac{1}{36}}$.
18. $b^2\sqrt{8\,a^5b}+ab\sqrt{50\,a^3b^3}-a^2\sqrt{128\,ab^5}$.
19. $\sqrt{3\,x^3+12\,x^2+12\,x}+\sqrt{27\,x^3-72\,x^2+48\,x}$.
20. $2\sqrt{12\,x^2+60\,xy+75\,y^2}-\sqrt{48\,x^2-72\,xy+27\,y^2}$.

TO REDUCE SURDS OF DIFFERENT DEGREES TO EQUIVALENT SURDS OF THE SAME DEGREE

209. *Ex.* Reduce $\sqrt{2}$, $\sqrt[3]{3}$, and $\sqrt[4]{5}$ to equivalent surds of the same degree.

By § 186,
$$\sqrt{2}=2^{\frac{1}{2}}=2^{\frac{6}{12}}=\sqrt[12]{2^6}=\sqrt[12]{64}.$$
$$\sqrt[3]{3}=3^{\frac{1}{3}}=3^{\frac{4}{12}}=\sqrt[12]{3^4}=\sqrt[12]{81}.$$
$$\sqrt[4]{5}=5^{\frac{1}{4}}=5^{\frac{3}{12}}=\sqrt[12]{5^3}=\sqrt[12]{125}.$$

Rule:

Express the surds with fractional exponents, reduce these to their lowest common denominator, and express the resulting expressions with radical signs.

The relative magnitudes of surds may be determined by reducing them, if necessary, to equivalent surds of the same degree.

Thus, in the above example, $\sqrt[12]{125}$ is greater than $\sqrt[12]{81}$, and $\sqrt[12]{81}$ than $\sqrt[12]{64}$.

Then, $\sqrt[4]{5}$ is greater than $\sqrt[3]{3}$, and $\sqrt[3]{3}$ than $\sqrt{2}$.

EXERCISE 87

Reduce the following to equivalent surds of the same degree:

1. $\sqrt{2}$ and $\sqrt[3]{3}$.
2. $\sqrt[3]{5}$ and $\sqrt[4]{7}$.
3. $\sqrt[3]{2}$ and $\sqrt[5]{3}$.
4. $\sqrt[4]{a}$, $\sqrt[7]{b}$, $\sqrt{c}$.
5. $\sqrt{a-b}$, $\sqrt[3]{a+b}$, $\sqrt[4]{a^2-b^2}$.
6. Is $\sqrt{2}$ greater than $\sqrt[3]{3}$?
7. Compare $\sqrt[5]{5}$ and $\sqrt[7]{7}$.
8. Write in order of magnitude $\sqrt[3]{4}$, $\sqrt[4]{6}$, $\sqrt[6]{15}$.
9. Which is greatest $\sqrt{3}$, $\sqrt[5]{15}$, $\sqrt[10]{253}$?
10. Which is greatest $\sqrt[3]{3}$, $\sqrt{2}$, $\sqrt[6]{4}$?

MULTIPLICATION AND EVOLUTION OF SURDS

210. 1. Multiply $\sqrt{6}$ by $\sqrt{15}$.

$$\sqrt{6}\times\sqrt{15}=\sqrt{6\times 15}=\sqrt{2\times 3\times 3\times 5}=\sqrt{3^2\times 2\times 5}=3\sqrt{10}.$$

2. Multiply $\sqrt{2a}$ by $\sqrt[3]{4a^2}$.

Reducing to equivalent surds of the same degree (§ 209),

$$\sqrt{2a}\times\sqrt[3]{4a^2}=(2a)^{\frac{1}{2}}\times(4a^2)^{\frac{1}{3}}=(2a)^{\frac{3}{6}}\times(4a^2)^{\frac{2}{6}}=\sqrt[6]{(2a)^3}\times\sqrt[6]{(4a^2)^2}$$
$$=\sqrt[6]{2^3a^3\times2^4a^4}=\sqrt[6]{2^6a^6\times2a}=2a\sqrt[6]{2a}.$$

We then have the following rule:

To multiply together two or more surds, reduce them, if necessary, to surds of the same degree.

Multiply together the expressions under the radical signs, and write the result under the common radical sign. *The result should be reduced to its simplest form.*

3. Multiply $\sqrt{5}$ by $\sqrt[6]{5}$.

By § 186, $\sqrt{5}=5^{\frac{1}{2}}=5^{\frac{3}{6}}=\sqrt[6]{5^3}$.

Then, $\sqrt{5}\times\sqrt[6]{5}=\sqrt[6]{5^3}\times\sqrt[6]{5}=\sqrt[6]{5^4}=5^{\frac{4}{6}}=5^{\frac{2}{3}}=\sqrt[3]{5^2}=\sqrt[3]{25}$.

4. Multiply $2\sqrt{3}+3\sqrt{2}$ by $3\sqrt{3}-\sqrt{2}$.

$$\begin{array}{r}
2\sqrt{3}+3\sqrt{2}\\
\underline{3\sqrt{3}-\sqrt{2}}\\
18+9\sqrt{6}\\
\underline{-2\sqrt{6}-6}\\
18+7\sqrt{6}-6=12+7\sqrt{6}.
\end{array}$$

To multiply a surd of the second degree by itself simply removes the radical sign; thus, $\sqrt{3}\times\sqrt{3}=3$, or $3^{\frac{1}{2}}\cdot3^{\frac{1}{2}}=3^{\frac{1}{2}+\frac{1}{2}}=3$.

5. Multiply $3\sqrt{1+x}-4\sqrt{x}$ by $\sqrt{1+x}+2\sqrt{x}$.

$$\begin{array}{l}
3\sqrt{1+x}-4\sqrt{x}\\
\underline{\sqrt{1+x}+2\sqrt{x}}\\
3\,(1+x)-4\sqrt{x+x^2}\\
\underline{+6\sqrt{x+x^2}-8x}\\
3\,(1+x)+2\sqrt{x+x^2}-8x=3-5x+2\sqrt{x+x^2}.
\end{array}$$

EXERCISE 88

Multiply the following:

1. $\sqrt{3}$ by $\sqrt{27}$.
2. $\sqrt[3]{36x^2y}$ by $\sqrt[3]{6xy^2}$.
3. $10^{\frac{1}{2}}$ by $30^{\frac{1}{2}}$.
4. $(108)^{\frac{1}{2}}$ by $(192)^{\frac{1}{2}}$.
5. $\sqrt[3]{72}$ by $\sqrt[3]{12}$.
6. $\sqrt{\frac{1}{2}}$ by $\sqrt{\frac{1}{8}}$.

7. $\sqrt{\frac{4}{14}}$ by $\sqrt{\frac{1}{8}}$.

8. $\sqrt{a^2-b^2}$ by $\sqrt{a+b}$.

9. $\sqrt[3]{3}$ by $\sqrt{2}$.

10. $\sqrt[4]{7}$ by $\sqrt{7}$.

11. $\sqrt[3]{7\,ab^2}$ by $\sqrt{7\,ab}$.

12. $\sqrt[3]{5\,a^2}$ by $\sqrt{125\,ab}$.

13. $2+3^{\frac{1}{2}}$ by $3+2^{\frac{1}{2}}$.

14. $5+\sqrt{7}\cdot 2+\sqrt{5}$.

15. $(5+\sqrt{7})(2+\sqrt{5})$.

16. Expand $(\sqrt{a}+\sqrt{b})^2$ (§ 91).

17. Expand $[(a)^{\frac{1}{2}}+7(b)^{\frac{1}{2}}][(a)^{\frac{1}{2}}-5(b)^{\frac{1}{2}}]$.

18. Expand $(\sqrt{3\,a}+\sqrt{4\,b})(\sqrt{3\,a}-\sqrt{4\,b})$.

19. Expand $[3\cdot 2^{\frac{1}{2}}-2\cdot 3^{\frac{1}{2}}]^2$.

20. Expand $(\sqrt{1}+\sqrt{2}+\sqrt{3})^2$.

21. Expand $(2\sqrt{3})^3$, $(3\sqrt[3]{2})^3$, $(4\sqrt[4]{2})^4$.

22. Expand $(\sqrt[6]{12})^3$.

$(\sqrt[6]{12})^3=(12)^{\frac{3}{6}}$ (§ 193) $=12^{\frac{1}{2}}=2\sqrt{3}$.

23. Expand $(\sqrt[6]{12})^{12}$.

24. Expand $(\sqrt[9]{54\,a^2b})^3$.

25. Expand $(4\sqrt{a-b}+3\sqrt{a+b})^2$.

Expand and express result in the form $a+2\sqrt{b}$:

26. $(5+\sqrt{3})^2=25+10\sqrt{3}+3=28+2\cdot 5\sqrt{3}=28+2\sqrt{75}$.

27. $(5^{\frac{1}{2}}-3^{\frac{1}{2}})^2$.

28. $(2^{\frac{1}{2}}+3^{\frac{1}{2}})^2$.

29. $(\sqrt{5}+\sqrt{7})^2$.

30. $(\sqrt{6}-2\sqrt{3})^2$.

31. $(7+4\sqrt{3})^2$.

32. $[3+(2)^{\frac{1}{2}}]^2$.

33. $(2\cdot 2^{\frac{1}{2}}+3\cdot 3^{\frac{1}{2}})^2$.

Supply the missing term in the following trinomial squares:

34. $4+2\sqrt{12}+$?

35. $7+2\sqrt{14}$.

36. $14+2(14)^{\frac{1}{2}}$.

37. Extract the square root of $7+2\sqrt{12}$.

38. Extract the square root of $5+2(6)^{\frac{1}{2}}$.

Note that in squaring a binomial, one or both of whose terms are affected by the exponent $\frac{1}{2}$, the square reduces to a binomial surd if both terms of the binomial to be squared are numerical. (Compare Examples 27–36.) Also *the part* $2(\quad)^{\frac{1}{2}}$ *corresponds to the middle term of the examples in* § 91. *In* $2(ab)^{\frac{1}{2}}$ *if ab can be so factored that the sum of these factors is equal to the other term, the square roots of these factors connected by the sign of the irrational term will be the square root of the binomial surd.*

39. Find the square root of $8+2(15)^{\frac{1}{2}}$.

$$15=5\cdot 3 \text{ and } 5+3=8.$$

Hence by the above rule,

$$\sqrt{8+2(15)^{\frac{1}{2}}}=\sqrt{5}+\sqrt{3}.$$

40. $$[17+6(8)^{\frac{1}{2}}]^{\frac{1}{2}}=[17+2(72)^{\frac{1}{2}}]^{\frac{1}{2}}$$
$$=[17+2(8\cdot 9)^{\frac{1}{2}}]^{\frac{1}{2}}$$
$$=9^{\frac{1}{2}}+8^{\frac{1}{2}}$$
$$=9^{\frac{1}{2}}+(2^2)^{\frac{1}{2}}\cdot 2^{\frac{1}{2}}$$
$$=3+2\sqrt{2}.$$

Find the square roots of the following binomial surds:

Remember that the coefficient of the radical must be 2.

41. $3-2\sqrt{2}$.
45. $23+2\cdot 132^{\frac{1}{2}}$.
49. $37-640^{\frac{1}{2}}$.
42. $11+2(30)^{\frac{1}{2}}$.
46. $29+2\cdot 54^{\frac{1}{2}}$.
50. $4+(15)^{\frac{1}{2}}$.
43. $14+6\sqrt{5}$.
47. $55+3\sqrt{24}$.
51. $5+\sqrt{21}$.
44. $24+2\cdot 140^{\frac{1}{2}}$.
48. $12-\sqrt{108}$.
52. $55-20\sqrt{6}$.
53. $44-4(72)^{\frac{1}{2}}$.
54. $53-\sqrt{600}$.

DIVISION OF MONOMIAL SURDS

211. $$\sqrt[n]{ab}=\sqrt[n]{a}\times\sqrt[n]{b}.$$

Whence, $$\frac{\sqrt[n]{ab}}{\sqrt[n]{a}}=\sqrt[n]{b}.$$

Rule:

To divide one monomial *surd* by another, reduce them, if necessary, to surds of the same degree.

Divide the expression under the radical sign in the dividend by the expression under the radical sign in the

divisor, and write the result under the common radical sign. *The result should be reduced to its simplest form.*

1. Divide $\sqrt[3]{405}$ by $\sqrt[3]{5}$.

We have, $$\frac{\sqrt[3]{405}}{\sqrt[3]{5}}=\sqrt[3]{\frac{405}{5}}=\sqrt[3]{81}=\sqrt[3]{27\times 3}=3\sqrt[3]{3}.$$

2. Divide $\sqrt[3]{4}$ by $\sqrt{6}$.

Reducing to surds of the same degree (§ 209),

$$\frac{\sqrt[3]{4}}{\sqrt{6}}=\frac{4^{\frac{1}{3}}}{6^{\frac{1}{2}}}=\frac{(2^2)^{\frac{2}{6}}}{(2\times 3)^{\frac{3}{6}}}=\frac{\sqrt[6]{2^4}}{\sqrt[6]{2^3\times 3^3}}=\sqrt[6]{\frac{2^4}{2^3\times 3^3}}=\sqrt[6]{\frac{2}{3^3}}=\sqrt[6]{\frac{2\times 3^3}{3^6}}=\frac{1}{3}\sqrt[6]{54}.$$

3. Divide $\sqrt{10}$ by $\sqrt[6]{40}$.

We have, $$\sqrt{10}=10^{\frac{1}{2}}=10^{\frac{3}{6}}=(10^3)^{\frac{1}{6}}=(5^3\cdot 2^3)^{\frac{1}{6}}.$$

Then, $$\frac{\sqrt{10}}{\sqrt[6]{40}}=\left(\frac{2^3\cdot 5^3}{2^3\cdot 5}\right)^{\frac{1}{6}}=(5^2)^{\frac{1}{6}}=5^{\frac{1}{3}}=\sqrt[3]{5}.$$

EXERCISE 89

Divide the following:

1. $\sqrt{60}$ by $\sqrt{3}$.
2. $(72)^{\frac{1}{2}}$ by $(2)^{\frac{1}{2}}$.
3. $\sqrt{48}$ by $\sqrt{32}$.
4. $75^{\frac{1}{2}}$ by $60^{\frac{1}{2}}$.
5. $6\cdot 3^{\frac{1}{2}}$ by $2\cdot 3^{\frac{1}{2}}$.
6. $\sqrt[3]{32}$ by $\sqrt[3]{4}$.
7. $45^{\frac{1}{3}}$ by $9^{\frac{1}{3}}$.
8. $\sqrt{128}$ by $\sqrt{48}$.
9. $\sqrt[3]{12}$ by $\sqrt{16}$.
10. $20\sqrt{12}$ by $8\sqrt{3}$.
11. $(6\,a^2b)^{\frac{1}{5}}$ by $(96\,bc^3)^{\frac{1}{10}}$.
12. $\sqrt[5]{3\,a^3}$ by $\sqrt{2\,a}$.
13. $\sqrt{27\,x^3}$ by $\sqrt[3]{36\,x^2}$.
14. $\sqrt[3]{\frac{49}{27}}$ by $\sqrt{\frac{7}{8}}$.
15. $(\frac{52}{15})^{\frac{1}{2}}$ by $(\frac{65}{12})^{\frac{1}{2}}$.
16. $\sqrt[5]{\frac{4}{9}}$ by $\sqrt{\frac{2}{3}}$.
17. $(\frac{21}{44})^{\frac{1}{2}}$ by $(\frac{28}{33})^{\frac{1}{2}}$.
18. $(\frac{12}{15})^{\frac{1}{2}}$ by $2\cdot 3^{\frac{1}{2}}$.

EVOLUTION OF SURDS

212. 1. Extract the cube root of $\sqrt[5]{27\,x^3}$.

$$\sqrt[3]{(\sqrt[5]{27\,x^3})}=(\sqrt[5]{(3\,x)^3})^{\frac{1}{3}}=[(3\,x)^{\frac{3}{5}}]^{\frac{1}{3}}=(3\,x)^{\frac{1}{5}}=\sqrt[5]{3\,x}.$$

2. Extract the fifth root of $\sqrt{6}$.

$$\sqrt[5]{(\sqrt[3]{6})} = (6^{\frac{1}{3}})^{\frac{1}{5}} = 6^{\frac{1}{15}} = \sqrt[15]{6}.$$

Then, to extract any root of a surd,

If possible, extract the required root of the expression under the radical sign; otherwise, multiply the index of the surd by the index of the required root.

If the surd has a coefficient which is not a perfect power of the degree denoted by the index of the required root, it should be introduced under the radical sign (§ 206) before applying the rule.

Thus, $\sqrt[5]{(4\sqrt{2})} = \sqrt[5]{(\sqrt{32})} = \sqrt{2}$,

or $\sqrt[5]{(4\sqrt{2})} = (4 \cdot 2^{\frac{1}{2}})^{\frac{1}{5}} = (2^2 \cdot 2^{\frac{1}{2}})^{\frac{1}{5}} = 2^{\frac{2}{5}} \cdot 2^{\frac{1}{10}} = 2^{\frac{4}{10}} \cdot 2^{\frac{1}{10}} = 2^{\frac{5}{10}} = \sqrt{2}$.

EXERCISE 90

Find the values of the following:

1. $\sqrt{(\sqrt[6]{25})}$.
2. $\sqrt[3]{(\sqrt[7]{8a^3b^6})}$.
3. $\sqrt{(\sqrt[3]{13})}$.
4. $\sqrt[5]{(\sqrt[4]{243x^5})}$.
5. $\sqrt{(\sqrt[5]{9a^2+12a+4})}$.
6. $\sqrt[3]{(\sqrt[4]{49})}$.
7. $\sqrt[4]{(81\sqrt[3]{16})}$.
8. $\sqrt[5]{(2\sqrt[3]{3a^2b})}$.
9. $\sqrt[4]{(16a^4\sqrt[4]{3a})}$.
10. $\sqrt[7]{(2x\sqrt[5]{4x^2})}$.
11. $\sqrt[6]{(\sqrt{343})}$.
12. $\sqrt[3]{(2n^2\sqrt[5]{16n^2})}$.

REDUCTION OF A FRACTION WHOSE DENOMINATOR IS IRRATIONAL (§ 198) TO AN EQUIVALENT FRACTION HAVING A RATIONAL DENOMINATOR

213. Case I. *When the denominator is a monomial.*

The reduction may be effected by multiplying both terms of the fraction by a surd of the same degree as the denominator, having under its radical sign such an expression as will make the denominator of the resulting fraction rational.

Ex. Reduce $\dfrac{5}{\sqrt[3]{3a^2}}$ to an equivalent fraction having a rational denominator.

Multiplying both terms by $\sqrt[3]{9a}$, we have

$$\frac{5}{\sqrt[3]{3a^2}} = \frac{5\sqrt[3]{9a}}{\sqrt[3]{3a^2}\sqrt[3]{9a}} = \frac{5\sqrt[3]{9a}}{\sqrt[3]{27a^3}} = \frac{5\sqrt[3]{9a}}{3a}.$$

EXERCISE 91

Reduce each of the following to an equivalent fraction having a rational denominator:

1. $\frac{2}{\sqrt{3}}$.
2. $\frac{1}{\sqrt{2}}$.
3. $\sqrt{\frac{5}{2}}$.
4. $\frac{5}{\sqrt{7}}$.
5. $\frac{3}{\sqrt[3]{4}}$.
6. $\frac{a^2}{\sqrt[3]{9a^2}}$.
7. $\frac{9b}{\sqrt[3]{36b}}$
8. $\frac{9}{\sqrt[5]{27}}$.

214. Case II. *When the denominator is a binomial containing only surds of the second degree.*

1. Reduce $\frac{5-\sqrt{2}}{5+\sqrt{2}}$ to an equivalent fraction having a rational denominator.

Multiplying both terms by $5-\sqrt{2}$ ($5-\sqrt{2}$ is called the conjugate of $5+\sqrt{2}$), we have

$$\frac{5-\sqrt{2}}{5+\sqrt{2}}=\frac{(5-\sqrt{2})^2}{(5+\sqrt{2})(5-\sqrt{2})}=\frac{25-10\sqrt{2}+2}{25-2}\ (\S\S\ 89,\ 91)\ =\frac{27-10\sqrt{2}}{23}.$$

2. Reduce $\frac{3\sqrt{a}-2\sqrt{a-b}}{2\sqrt{a}-3\sqrt{a-b}}$ to an equivalent fraction having a rational denominator.

Multiplying both terms by $2\sqrt{a}+3\sqrt{a-b}$,

$$\frac{3\sqrt{a}-2\sqrt{a-b}}{2\sqrt{a}-3\sqrt{a-b}}=\frac{(3\sqrt{a}-2\sqrt{a-b})(2\sqrt{a}+3\sqrt{a-b})}{(2\sqrt{a}-3\sqrt{a-b})(2\sqrt{a}+3\sqrt{a-b})}$$
$$=\frac{6a+5\sqrt{a}\sqrt{a-b}-6(a-b)}{4a-9(a-b)}=\frac{6b+5\sqrt{a^2-ab}}{9b-5a}.$$

Rule: — Multiply both numerator and denominator of the fraction by the denominator with the sign between its terms changed.

EXERCISE 92

Reduce each of the following to an equivalent fraction having a rational denominator:

1. $\frac{3}{\sqrt{3}+\sqrt{2}}$.
2. $\frac{8}{3-(5)^{\frac{1}{2}}}$.
3. $\frac{4-(2)^{\frac{1}{2}}}{4+(2)^{\frac{1}{2}}}$.

4. $\dfrac{a+\sqrt{b}}{a-\sqrt{b}}$.

5. $\dfrac{\sqrt{10}-6\sqrt{2}}{\sqrt{10}+3\sqrt{2}}$.

6. $\dfrac{(x^2+a^2)^{\frac{1}{2}}+a}{(x^2+a^2)^{\frac{1}{2}}-a}$.

7. $1+\dfrac{a^2}{\sqrt{x^2-a^2}}$.

8. $\dfrac{x^{\frac{1}{2}}-(x+y)^{\frac{1}{2}}}{x^{\frac{1}{2}}+(x+y)^{\frac{1}{2}}}$.

9. $\dfrac{2+\sqrt{\frac{2}{x}}}{2-\sqrt{\frac{2}{x}}}$.

10. $\dfrac{3-(a-3)^{\frac{1}{2}}}{3+(a-3)^{\frac{1}{2}}}$.

11. $\dfrac{5\cdot 2^{\frac{1}{2}}+6^{\frac{1}{2}}}{3\cdot 2^{\frac{1}{2}}-6^{\frac{1}{2}}}$.

12. $\dfrac{\sqrt{x-2}+1}{\sqrt{x-2}+2}$.

Add the following fractions:

(The common denominator is more readily found if the denominators of the fractions are first rationalized.)

13. $\dfrac{2}{3^{\frac{1}{2}}+2^{\frac{1}{2}}}+\dfrac{3}{3^{\frac{1}{2}}-2^{\frac{1}{2}}}$.

14. $\dfrac{2\sqrt{6}+1}{\sqrt{3}+2}+\dfrac{5+\sqrt{6}}{\sqrt{12}}$.

15. $\dfrac{a+b^{\frac{1}{2}}}{a-2\,b^{\frac{1}{2}}}-\dfrac{a-b^{\frac{1}{2}}}{a+b^{\frac{1}{2}}}$.

16. $\dfrac{(x+1)^{\frac{1}{2}}+2}{(x+1)^{\frac{1}{2}}-2}+\dfrac{3\sqrt{2}}{(2\,x+2)^{\frac{1}{2}}}$.

215. The approximate value of a fraction whose denominator is irrational may be conveniently found by reducing it to an equivalent fraction with a rational denominator.

Ex. Find the approximate value of $\dfrac{1}{2-\sqrt{2}}$ to three places of decimals.

$$\frac{1}{2-\sqrt{2}}=\frac{2+\sqrt{2}}{(2-\sqrt{2})(2+\sqrt{2})}=\frac{2+\sqrt{2}}{4-2}=\frac{2+1.414\cdots}{2}=1.707\cdots.$$

The $\sqrt{2}$ and the $\sqrt{3}$ are important values and are of frequent occurrence in mathematical investigation.

EXERCISE 93

Find the values of the following to three places of decimals:

1. $\dfrac{3}{\sqrt{5}}$.

2. $\dfrac{1}{\sqrt{2}}$.

3. $\dfrac{1}{\sqrt{3}}$.

4. $\dfrac{2}{3-\sqrt{3}}$.

5. $\dfrac{1}{5+2\sqrt{7}}$.

6. $\dfrac{2}{\sqrt{5}-1}$.

7. $\dfrac{5}{2\sqrt{3}-4}$.

8. $\dfrac{3}{\sqrt{10}}$.

9. $\dfrac{1}{\sqrt{6}-2}$.

216. Important Property of Quadratic Surds (§ 201). I. *A quadratic surd cannot equal the sum of a rational expression and a quadratic surd.*

For, if possible, let $\sqrt{a}=b+\sqrt{c}$,

where b is a rational expression, and $\sqrt{a}$ and $\sqrt{c}$ quadratic surds.

Squaring both members, $a=b^2+2b\sqrt{c}+c$,

or,
$$2b\sqrt{c}=a-b^2-c.$$

Whence,
$$\sqrt{c}=\frac{a-b^2-c}{2b}.$$

That is, a quadratic surd equal to a rational expression.

But this is impossible; whence, $\sqrt{a}$ cannot equal $b+\sqrt{c}$.

II. *If $a+\sqrt{b}=c+\sqrt{d}$, where a and c are rational expressions, and $\sqrt{b}$ and $\sqrt{d}$ quadratic surds, then*

$$a=c, \text{ and } \sqrt{b}=\sqrt{d}.$$

If a does not equal c, let $a=c+x$; then, x is rational.

Substituting this value in the given equation,

$$c+x+\sqrt{b}=c+\sqrt{d}, \text{ or } x+\sqrt{b}=\sqrt{d}.$$

But this is impossible by § 216.

Then, $a=c$, and therefore $\sqrt{b}=\sqrt{d}$.

That is, an equation of the form $a+\sqrt{b}=c+\sqrt{d}$ may be written as two equations,

$$a=c, \quad b=d.$$

217. Solution of Equations having the Unknown Numbers under Radical Signs.

1. Solve the equation $(x^2-5)^{\frac{1}{2}}-x=-1$.

Transposing $-x$, $(x^2-5)^{\frac{1}{2}}=x-1$.

Squaring both members, $x^2-5=x^2-2x+1$.

Transposing, $2x=6$; whence, $x=3$.

(Substituting 3 for x in the given first member, and taking the positive value of the square root, the first member becomes

$$(9-5)^{\frac{1}{2}}-3=2-3=-1;$$

which shows that the solution $x=3$ is correct.)

Where no sign occurs before a radical the positive sign is understood.

Also, in equations of the type, $\sqrt{x+3}+\sqrt{x^2+9}-\sqrt{x-3}=\sqrt{x}$, usage requires that we regard only the given sign before the radical rather than the double sign that naturally belongs to a radical.

Rule: — **Transpose the terms of the equation so that a surd term may stand alone in one member; then raise both members to a power of the same degree as the surd.**

If surd terms still remain, repeat the operation.

The equation should be simplified as much as possible before performing the involution.

2. Solve the equation $\sqrt{2x-1}+\sqrt{2x+6}=7$.

Transposing $\sqrt{2x-1}$, $\sqrt{2x+6}=7-\sqrt{2x-1}$.

Squaring, $2x+6=49-14\sqrt{2x-1}+2x-1$.

Transposing, $14\sqrt{2x-1}=42$, or $\sqrt{2x-1}=3$.

Squaring, $2x-1=9$; whence, $x=5$.

Substitute $x=5$ in the given equation to verify the result.

EXERCISE 94

Solve the following equations, verifying each result:

(Any radical may be changed to a form with fractional exponents and the exponent form is often more easily solved.)

1. $(2x+1)^{\frac{1}{2}}-5=0$.
2. $\sqrt{2x+7}+5=8$.
3. $(4t^2-19)^{\frac{1}{2}}=2t-1$.
4. $\sqrt{u^2-11}+u=11$.
5. $\dfrac{2}{(z+8)^{\frac{1}{2}}}-\dfrac{2z}{(2-2z)^{\frac{1}{2}}}=(2-2z)^{\frac{1}{2}}$.
6. $\sqrt{5m-24}+4=\sqrt{5m}$.
7. $(v)^{\frac{1}{2}}+(v-3)^{\frac{1}{2}}=3$.

8. $\dfrac{(3\,x+24)^{\frac{1}{2}}-(3\,x)^{\frac{1}{2}}}{(3\,x+24)^{\frac{1}{2}}+(3\,x)^{\frac{1}{2}}}=\dfrac{1}{2}$ (§ 147).

9. $\sqrt{u-3}-\sqrt{u+21}=-2\sqrt{u}$.

10. $(8\,x^3+36\,x^2)^{\frac{1}{3}}-3=2\,x$.

11. $\dfrac{\sqrt{x+a}+\sqrt{x-a}}{\sqrt{x+a}-\sqrt{x-a}}=2$. (Rationalize denominator, or use § 147.)

12. $(t^2-5\,t-2)^{\frac{1}{2}}+(t^2+3\,t+6)^{\frac{1}{2}}=4$.

13. $\sqrt{x+15}-\sqrt{x+3}=2\sqrt{x}$.

14. $\dfrac{(2\,x+8)^{\frac{1}{2}}+(2\,x)^{\frac{1}{2}}}{(2\,x+8)^{\frac{1}{2}}-(2\,x)^{\frac{1}{2}}}=2$.

15. $\sqrt{x-2\,a}-\sqrt{x-6\,a}=2\sqrt{x-5\,a}$.

IMAGINARY NUMBERS

218. If a number involves an indicated even root of a negative number it is called *imaginary*. Such numbers depend upon a new unit, $\sqrt{-1}$ or $(-1)^{\frac{1}{2}}$; as $\sqrt{-2}$, $\sqrt[4]{-3}$.

In contradistinction, rational and irrational numbers (§ 199) are called *real* numbers.

219. An imaginary number of the form $\sqrt{-a}$ is called a *pure imaginary* number, and the sum of a real and an imaginary is called a *complex* number; as $a+b\sqrt{-1}$.

220. Meaning of a Pure Imaginary Number.

If $\sqrt{a}$ is *real* (§ 218), we define $\sqrt{a}$ as an expression such that, when raised to the second power, the result is a (§ 165).

To find what meaning to attach to a pure imaginary number, we assume the above principle to hold when $\sqrt{a}$ is imaginary.

Thus, $\sqrt{-2}$ means an expression such that, when raised to the second power, the result is -2; that is, $(\sqrt{-2})^2$ or $(-2^{\frac{1}{2}})^2=-2$.

In like manner, $(\sqrt{-1})^2=(-1^{\frac{1}{2}})^2=-1$; etc.

OPERATIONS WITH IMAGINARY NUMBERS

221. By § 220, $(\sqrt{-5})^2=(-5^{\frac{1}{2}})^2=-5.$ (1)

Also, $(\sqrt{5}\sqrt{-1})^2=(\sqrt{5})^2(\sqrt{-1})^2=5(-1)=-5,$ (2)

or $(\sqrt{-5})^2=(5^{\frac{1}{2}})^2\cdot(-1^{\frac{1}{2}})^2=5(-1)=-5.$

From (1) and (2), $(\sqrt{-5})^2=(\sqrt{5}\sqrt{-1})^2.$

Whence, $\sqrt{-5}=\sqrt{5}\sqrt{-1}$, or $5^{\frac{1}{2}}(-1)^{\frac{1}{2}}.$

Then, *every imaginary square root can be expressed as the product of a real number by* $\sqrt{-1}$. *It is advisable to reduce every imaginary to this form before performing the indicated operations.*

$\sqrt{-1}$ is called the *imaginary unit;* it is often represented by i.

222. Addition and Subtraction of Imaginary Numbers.

Pure imaginary numbers may be added and subtracted in the same manner as surds.

1. Add $\sqrt{-4}$ and $\sqrt{-36}$.

By § 221, $\sqrt{-4}+\sqrt{-36}=2(-1)^{\frac{1}{2}}+6(-1)^{\frac{1}{2}}=8(-1)^{\frac{1}{2}}.$

2. Subtract $3-\sqrt{-9}$ from $1+\sqrt{-16}$.

In adding or subtracting complex numbers, we assume that the rules for adding or subtracting real numbers may be applied without change.

Then, $$1+\sqrt{-16}-(3-\sqrt{-9})=1+4\sqrt{-1}-3+3\sqrt{-1}$$
$$=-2+7\sqrt{-1}.$$

EXERCISE 95

Simplify the following:

1. $(-9)^{\frac{1}{2}}+(-25)^{\frac{1}{2}}.$
2. $\sqrt{-5}+\sqrt{-45}.$
3. $\sqrt{-18}-\sqrt{-8}.$
4. $\sqrt{-75}+\sqrt{-48}.$
5. $\sqrt{-(x+2)^2}-\sqrt{-x^2}.$
6. $(-x^2)^{\frac{1}{2}}+(-y^2)^{\frac{1}{2}}+(-z^2)^{\frac{1}{2}}.$
7. $3\sqrt{-36}+2\sqrt{-144}+\sqrt{-81}.$
8. $2(-16)^{\frac{1}{2}}-5(-49)^{\frac{1}{2}}-8(-121)^{\frac{1}{2}}.$
9. $\sqrt{-16\,x^2}-\sqrt{-9\,x^2}-\sqrt{-4\,x^2}.$

10. $\sqrt{-4a^2-4ab-b^2}-\sqrt{-9a^2+6ab-b^2}$.

11. Add $2+(-3)^{\frac{1}{2}}$ to $5+(-27)^{\frac{1}{2}}$.

12. Add $6-\sqrt{-64}$ to $1-\sqrt{-49}$.

13. From $2+\sqrt{-9}$ take $8-\sqrt{-25}$.

223. Positive Integral Powers of $\sqrt{-1}$ or $-1^{\frac{1}{2}}$.

By § 220, $(-1^{\frac{1}{2}})^2=(-1)^{\frac{1}{2}}\cdot(-1)^{\frac{1}{2}}=-1$. (By adding exponents.)

Then,
$$(\sqrt{-1})^3=(-1^{\frac{1}{2}})^2\cdot(-1)^{\frac{1}{2}}=-1\cdot(-1)^{\frac{1}{2}}=-\sqrt{-1};$$
$$(\sqrt{-1})^4=(-1^{\frac{1}{2}})^2\cdot(-1^{\frac{1}{2}})^2=-1\cdot-1 \quad =1;$$
$$(\sqrt{-1})^5=(-1^{\frac{1}{2}})^4\cdot(-1)^{\frac{1}{2}}= \quad 1\cdot-1^{\frac{1}{2}}=\sqrt{-1}, \text{ etc.}$$

Thus, the first four positive integral powers of $\sqrt{-1}$ are $\sqrt{-1}$, -1, $-\sqrt{-1}$, and 1; and for higher powers these terms recur in the same order, the sixth power being like the second, etc.

224. Multiplication of Imaginary Numbers.

The product of two or more imaginary square roots can be obtained by aid of the principles of §§ 221 and 223.

1. Multiply $\sqrt{-2}$ by $\sqrt{-3}$.

By § 221, $-2^{\frac{1}{2}}\cdot-3^{\frac{1}{2}}=2^{\frac{1}{2}}\cdot-1^{\frac{1}{2}}\cdot3^{\frac{1}{2}}\cdot-1^{\frac{1}{2}}$
$$=2^{\frac{1}{2}}\cdot3^{\frac{1}{2}}\cdot(-1^{\frac{1}{2}})^2=6^{\frac{1}{2}}(-1)\ (\S\ 223)=-\sqrt{6}.$$

2. Find the product of $\sqrt{-9}$, $\sqrt{-16}$, and $\sqrt{-25}$.

$$\sqrt{-9}\times\sqrt{-16}\times\sqrt{-25}=3\sqrt{-1}\times4\sqrt{-1}\times5\sqrt{-1}$$
$$=60(\sqrt{-1})^3=60(-\sqrt{-1})\ (\S\ 223)=-60\sqrt{-1}.$$

3. Multiply $2+5\sqrt{-5}$ by $4-3\sqrt{-5}$.

In multiplying complex numbers, we assume that the rules for multiplying real numbers may be applied without change.

$$\begin{array}{l}
2+\ \ 5\sqrt{5}\sqrt{-1}\\
\underline{4-\ \ 3\sqrt{5}\sqrt{-1}}\\
8+20\sqrt{5}\sqrt{-1}\\
\underline{\quad-\ \ 6\sqrt{5}\sqrt{-1}-15(5)(-1)}\\
8+14\sqrt{-5} \qquad +75=83+14\sqrt{-5}
\end{array}$$

4. Expand $(\sqrt{-5}+2\sqrt{-3})^2$ by the rule of § 91.

$$(-5^{\frac{1}{2}}+2\cdot -3^{\frac{1}{2}})^2=(5^{\frac{1}{2}}\cdot -1^{\frac{1}{2}}+2\cdot 3^{\frac{1}{2}}\cdot -1^{\frac{1}{2}})^2$$
$$=(5^{\frac{1}{2}})^2\cdot(-1^{\frac{1}{2}})^2+4\cdot 5^{\frac{1}{2}}\cdot -1^{\frac{1}{2}}\cdot 3^{\frac{1}{2}}\cdot -1^{\frac{1}{2}}+2^2\cdot(3^{\frac{1}{2}})^2\cdot(-1^{\frac{1}{2}})^2$$
$$=-5+4\cdot 15^{\frac{1}{2}}\cdot(-1^{\frac{1}{2}})^2+4\cdot -3$$
$$=-5-4\cdot 15^{\frac{1}{2}}-12=-17-4\sqrt{15}.$$

EXERCISE 96

Multiply the following:

1. $(-4)^{\frac{1}{2}}$ by $(-9)^{\frac{1}{2}}$.
2. $(-36)^{\frac{1}{2}}$ by $(-16)^{\frac{1}{2}}$.
3. $\sqrt{-8}$ by $\sqrt{-6}$.
4. $(-196\,a^2)^{\frac{1}{2}}$ by $(-144\,a^2)^{\frac{1}{2}}$.
5. $\sqrt{-16}$ by $\sqrt{-64}$.
6. $(-9)^{\frac{1}{2}}$ by $(-18)^{\frac{1}{2}}$.
7. $3+(-3)^{\frac{1}{2}}$ by $2+(-2)^{\frac{1}{2}}$.
8. $5+4\sqrt{-1}$ by $2-\sqrt{-3}$.
9. $3\sqrt{-x}+2\sqrt{-y}$ by $2\sqrt{-x}+3\sqrt{-y}$.
10. $8\sqrt{-7}$, $5\sqrt{-5}$, $-3\sqrt{-4}$, and $2\sqrt{-6}$.
11. $\sqrt{-16}$, $\sqrt{-49}$, $\sqrt{-64}$, and $\sqrt{-100}$.

Expand the following by inspection:

12. $[2+(-3)^{\frac{1}{2}}]^2$.
13. $(5-\sqrt{-7})^2$.
14. $(5\sqrt{-6}+3\sqrt{-3})^2$.
15. $[6+4(-3)^{\frac{1}{2}}][6-4(-3)^{\frac{1}{2}}]$.
16. $(\sqrt{-3\,x}+y)(\sqrt{-3\,x}-y)$.
17. $[(-5\,x)^{\frac{1}{2}}+7^{\frac{1}{2}}][(-5\,x)^{\frac{1}{2}}+5^{\frac{1}{2}}]$.
18. $(8\sqrt{-2}+3\sqrt{-5})(8\sqrt{-2}-3\sqrt{-5})$.
19. $(a+b\sqrt{-1})(a-b\sqrt{-1})$.
20. Add $(a+b\sqrt{-1})$ to $(a-b\sqrt{-1})$.

$a+b\sqrt{-1}$ and $a-b\sqrt{-1}$ are called CONJUGATE IMAGINARIES. Note that their sum and their product are *real*.

225. Division of Imaginary Numbers.

1. Divide $\sqrt{-40}$ by $\sqrt{-5}$.

By § 221, $\dfrac{-40^{\frac{1}{2}}}{-5^{\frac{1}{2}}}=\dfrac{40^{\frac{1}{2}}\cdot -1^{\frac{1}{2}}}{5^{\frac{1}{2}}\cdot -1^{\frac{1}{2}}}=\dfrac{40^{\frac{1}{2}}}{5^{\frac{1}{2}}}=8^{\frac{1}{2}}=2\sqrt{2}.$

2. Divide $\sqrt{15}$ by $\sqrt{-3}$.

$$\frac{\sqrt{15}}{\sqrt{-3}}=\frac{-\sqrt{15}(-1)}{\sqrt{3}\sqrt{-1}}=\frac{-\sqrt{15}(\sqrt{-1})^2}{\sqrt{3}\sqrt{-1}}(\S\ 223)=-\sqrt{5}\sqrt{-1}=-\sqrt{-5}.$$

3. Reduce $\dfrac{\sqrt{3}-\sqrt{-2}}{\sqrt{3}+\sqrt{-2}}$ to an equivalent fraction having a real denominator.

Multiply both numerator and denominator of the fraction by the conjugate of the denominator, i. e., $\sqrt{3}-\sqrt{-2}$, that is, the denominator with the sign between its terms changed.

$$\frac{\sqrt{3}-\sqrt{-2}}{\sqrt{3}+\sqrt{-2}}=\frac{(\sqrt{3}-\sqrt{-2})^2}{(\sqrt{3})^2-(\sqrt{-2})^2}\ (\S\,89)$$

$$=\frac{(\sqrt{3})^2-2\sqrt{3}\sqrt{-2}+(\sqrt{-2})^2}{3-(-2)}\ (\S\ 91)$$

$$=\frac{3-2\sqrt{-6}-2}{3+2}=\frac{1-2\sqrt{-6}}{5}.$$

EXERCISE 97

Divide the following:

1. $-20^{\frac{1}{2}}$ by $-5^{\frac{1}{2}}$.
2. $\sqrt{-18}$ by $\sqrt{-3}$.
3. $-36^{\frac{1}{2}}$ by $-12^{\frac{1}{2}}$.
4. $-(-12\,a^2b)^{\frac{1}{2}}$ by $(3\,ab)^{\frac{1}{2}}$.
5. $-\sqrt{32}$ by $-\sqrt{-8}$.
6. $(180)^{\frac{1}{2}}$ by $-(10)^{\frac{1}{2}}$.
7. $-\sqrt[4]{-18}$ by $\sqrt[4]{-2}$.
8. $-\sqrt{a}$ by $\sqrt{-a^2}$.

Reduce each of the following to an equivalent fraction having a real denominator:

9. $\dfrac{2}{2-\sqrt{-3}}$.
10. $\dfrac{2+(-3)^{\frac{1}{2}}}{2-(-3)^{\frac{1}{2}}}$.
11. $\dfrac{3\sqrt{-a}+2\sqrt{-b}}{3\sqrt{-a}-2\sqrt{-b}}$.
12. $\dfrac{2\sqrt{-5}-5}{2\sqrt{-5}+5}$.

XIV. QUADRATIC EQUATIONS

226. A **Quadratic Equation** is an equation of the second degree (§ 75), with one or more unknown numbers.

A **Pure Quadratic Equation** is a quadratic equation involving only the square of the unknown number; as, $2x^2=5$.

An **Affected Quadratic Equation** is a quadratic equation involving both the square and the first power of the unknown number; as, $2x^2-3x-5=0$.

In § 103, we showed how to solve quadratic equations of the forms

$$ax^2+bx=0,\ ax^2+c=0,\ x^2+ax+b=0, \text{ and } ax^2+bx+c=0,$$

when the first members could be resolved into factors.

PURE QUADRATIC EQUATIONS

227. Let it be required to solve the equation

$$x^2-4=0,$$

$$\text{or} \quad x^2=4.$$

Taking the square root of each member, we have

$$\pm x=\pm 2;$$

for the square root of a number may be either + or − (§ 168).

But the equations $-x=2$ and $x=-2$ are the same as $x=-2$ and $x=2$, respectively, with all signs changed.

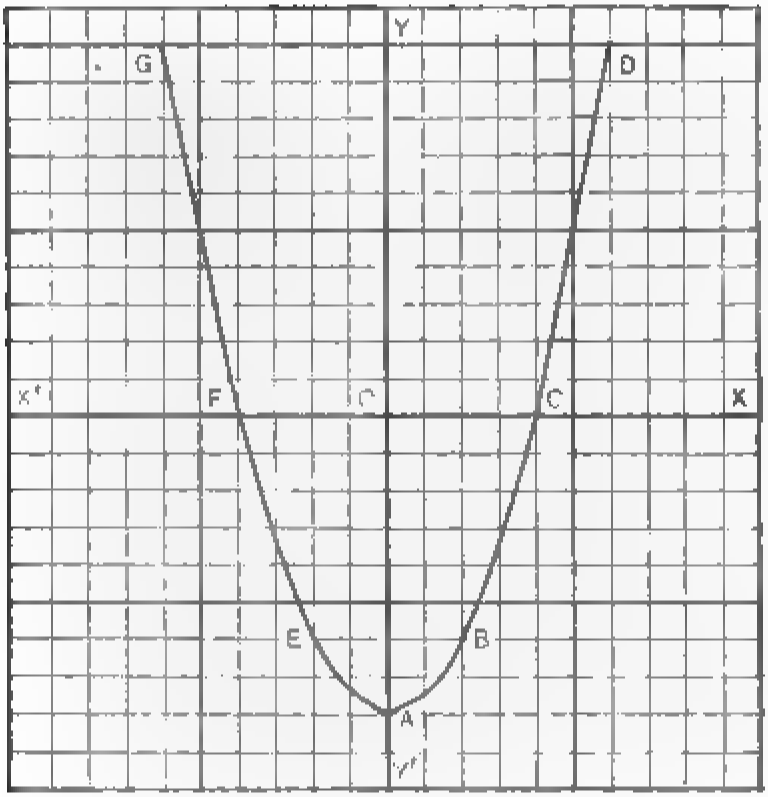
We then get all the values of x by equating the *positive* square root of the first member to the $\pm$ square root of the second.

The graph of a quadratic expression, with one unknown number, x, may be found by putting y equal to the expression, and finding the graph of the resulting equation as in § 151.

In the equation $\qquad x^2-4=0 \qquad$ placing

$$y=x^2-4,$$

that is, substituting $y=0$, and finding values for y by assigning values 0, 1, 2, etc., to x, we have



x	y	
0	-4	(A)
1	-3	(B)
2	0	(C)
3	5	(D)
4	12	
-1	-3	(E)
-2	-0	(F)
-3	5	(G)
-4	12	

Connecting these points (A), (B), (C), etc., we find, that they form a smooth curve, that the lowest point of the curve is on the y-axis, that the curve crosses the x-axis at ± 2. That is, the curve $x^2-4=0$ crosses the line $y=0$ (the x-axis) in two points and that these points of intersection $(2, 0)$, $(-2, 0)$ correspond to the algebraic solution of $x^2-4=0$ in § 227.

In general, it will be found that the graph of every equation of the second degree (§ 75) in two variables (unknown numbers) is a curve. The above geometrical picture shows in a graphical way why a quadratic equation has two roots.

EXERCISE 98

Find the graph of:

1. $x^2-9=0$. 2. $x^2-16=0$. 3. $x^2-25=0$.

228. A pure quadratic equation may be solved by reducing it, if necessary, to the form $x^2=a$, and then equating x to $\pm$ the square root of a (§ 227).

1. Solve the equation $3x^2+7=\dfrac{5x^2}{4}+35$.

Clearing of fractions, $12x^2+28=5x^2+140$.

Transposing, and uniting terms, $7x^2=112$, or $x^2=16$.

Equating x to the $\pm$ square root of 16, $x=\pm 4$.

Verify by substituting $x=\pm 4$ in the given equation.

2. Solve the equation $7x^2-5=5x^2-13$.

Transposing, and uniting terms, $2x^2=-8$, or $x^2=-4$.

Equating x to the $\pm$ square root of -4, $x=\pm\sqrt{-4}$

$$=\pm 2\sqrt{-1} \text{ (§ 221).}$$

In this case, both values of x are *imaginary* (§ 219); it is impossible to find a real value of x which will satisfy the given equation.

Verify by substituting in the given equation.

Make a graph of $x^2=-4$.

In solving fractional quadratic equations, any solution which does not satisfy the given equation must be rejected.

Thus, let it be required to solve the equation

$$\frac{x^2-7}{x^2+x-2}=\frac{1}{x+2}-\frac{1}{x-1}.$$

Multiplying both members by $(x+2)(x-1)$, or x^2+x-2,

$$x^2-7=x-1-x-2, \text{ or } x^2=4.$$

Extracting square roots, $x=\pm 2$.

The solution $x=-2$ does not satisfy the given equation; the only solution is $x=2$.

EXERCISE 99

Solve the following equations and verify each result:

1. $8v^2-24=7v^2+25$.

2. $\dfrac{5}{3x^2}-\dfrac{4}{5x^2}=\dfrac{13}{60}$.

3. $3(2t+4)^2+4(3t-2)^2=256$.

4. $\dfrac{4x}{3}+\dfrac{2}{3x}=\dfrac{7x}{12}+\dfrac{2}{x}$.

5. $\dfrac{2x^2+4}{5}-\dfrac{3x^2-7}{3}=\dfrac{11}{15}$.

6. $\dfrac{3x^2-4}{12}-\dfrac{2x^2+2}{5x^2+4}=\dfrac{x^2-3}{4}$. (§ 132, Ex 3.)

7. $\sqrt{10+u}-\sqrt{10-u}=2$.

8. $\dfrac{4v^2+3}{7}-\dfrac{8v^2-1}{2}=\dfrac{1}{14}$.

9. $\dfrac{3a}{t-5b}-\dfrac{t+5b}{3a+10b}=0$. Solve for t.

10. $\dfrac{4x^2-1}{4}+\dfrac{3x^2-4}{3}=\dfrac{2x^2-3}{2}$.

11. $\dfrac{t+a}{t-a}+\dfrac{t-a}{t+a}=\dfrac{a+c}{a-c}+\dfrac{a-c}{a+c}.$ (§ 133, Exs. 1, 2.)

12. $\sqrt{x^2+2}=-x-\dfrac{4}{\sqrt{x^2+2}}.$

The following equations occur in the study of physics.

Solve in the first six equations for the number which appears to the second power.

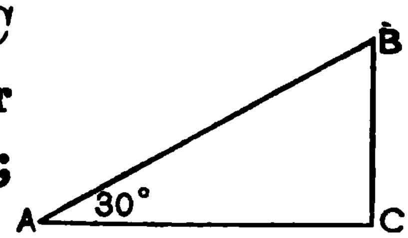
13. $S=\frac{1}{2}gt^2.$ 15. $F=\dfrac{mM}{d^2}.$ 17. $f=\dfrac{mv^2}{R}.$

14. $E=\frac{1}{2}mv^2.$ 16. $H=C^2Rt.$ 18. $R=\dfrac{kl}{D^2}.$

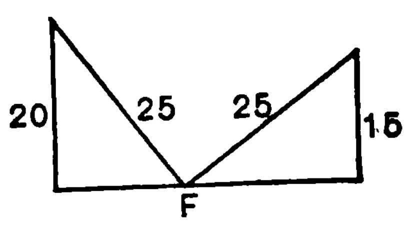
19. If the square of a certain number divided by 4 is added to twice the square of the number divided by 32, the sum is 20; find the number.

20. One number is five times another, and the difference of their squares is 216; find the numbers.

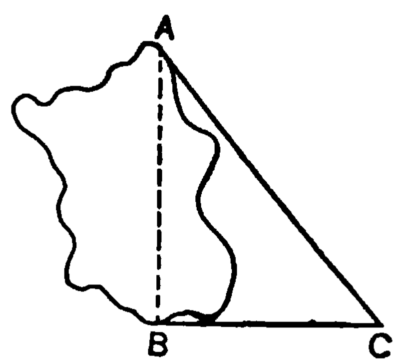
21. If one angle of the right triangle ABC is 30°, the hypothenuse is twice the shorter side. The side opposite angle B is $10\sqrt{3}$; find CB and explain your negative root.



22. A ladder 25 feet long standing in a court, will reach a window on one side of the court 20 feet from the ground. If turned on its foot as an axis it will reach a window in the opposite wall 15 feet from the ground. Find distance across the court and explain your negative roots.

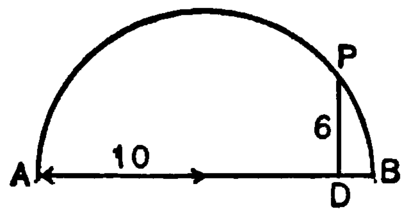


23. Two camps, A and B, are at opposite sides of a lake. In order to find the distance between them, a line BC was measured at right angles to AB. BC was found to be 441 feet. AC was measured and found to be 735 feet. Find AB.



24. In a semicircle, if the perpendicular DP be dropped from a point P in the circumference to the diameter AB, DP

is a mean proportional between the segments AD and DB. If the perpendicular DP is 6, find AD and DB, the radius of the circle being 10. If DP is 10 and the radius 6, what effect does it have on your solution? Draw the figure.



25. When a body falls from rest from any point above the earth's surface, the distance, S, which it traverses in any number of seconds, t, is found to be given by the equation

$$S = \frac{1}{2} gt^2,$$

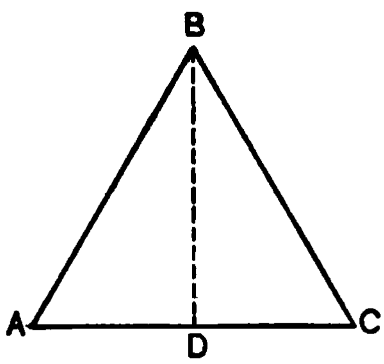
in which g represents the velocity which the body acquires in one second; $g = 32.15$ feet, or 980 centimeters.

A stone fell from a balloon a mile high; how much time elapsed before it reached the earth?

26. In the equation $t = \pi\sqrt{\frac{l}{g}}$, t represents the time required by a pendulum to make one vibration, l represents the length of the pendulum, and g is the same as in Problem 25. Find the length of a pendulum which beats seconds.

27. If a pendulum which beats seconds is found to be 99.3 centimeters long, find from the above equation the value of g.

28. The area of an equilateral triangle ABC is $16\sqrt{3}$. Find the altitude DB.

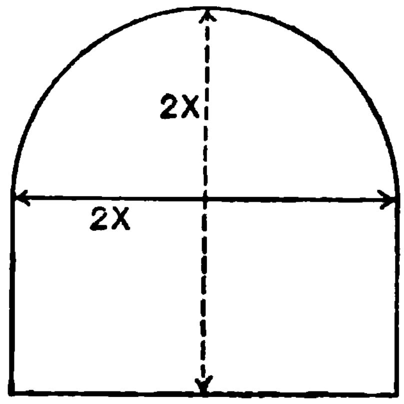


29. Two balloons start at the same time from St. Louis on a long-distance race. One strikes a northwest current carrying it 30 miles an hour; the other strikes a southwest current carrying it 25 miles an hour. At the end of the second hour each balloon is one mile from the earth. How far apart are they?

30. Two automobiles start from A at the same time, one going north at 18 miles an hour, the other going east at 15

miles an hour. How far apart are they at the end of the first hour?

31. D is due west of C, A is due north of D, and the distance from C to D is 84 miles. At 2 P. M. a train leaves C for D, running 40 miles an hour. At 2:30 P. M. a train leaves D for A, running 44 miles an hour. How far apart are they at 3 P. M.?



32. A window in the form of a rectangle surmounted by a semicircle is found to admit the most light when its height and width are equal. If the area of this window is 32.1372, find the width.

AFFECTED QUADRATIC EQUATIONS

What must be added to x^2+4x to form a perfect trinomial square? (Exercise 30.) What must be added to x^2+10x to form a perfect trinomial square?

229. First Method of Completing the Square.

By transposing the terms involving x to the first member, and all other terms to the second, and then dividing both members by the coefficient of x^2, any affected quadratic equation can be reduced to the form $x^2+px=q$.

We then add to both members such an expression as will make the first member a trinomial perfect square (Exercise 30); an operation which is termed *completing the square.*

Ex. Solve the equation $x^2+3x=4$.

A trinomial is a perfect square when its first and third terms are perfect squares and positive, and its second term plus or minus twice the product of their square roots (Exercise 30).

Then, *the square root of the third term is equal to the second term divided by twice the square root of the first.*

Hence, the *square root* of the expression which must be added to x^2+3x to make it a perfect square is $3x \div 2x$, or $\frac{3}{2}$.

Adding to both members the square of $\frac{3}{2}$, we have

$$x^2+3x+(\tfrac{3}{2})^2=4+\tfrac{9}{4}=\tfrac{25}{4}.$$

Equating the square root of the first member to the $\pm$ square root of the second (compare § 227), we have

$$x+\tfrac{3}{2}=\pm\tfrac{5}{2}.$$

Transposing $\frac{3}{2}$, $\quad x=-\frac{3}{2}+\frac{5}{2}$ or $-\frac{3}{2}-\frac{5}{2}=1$ or -4.

Rule:

Reduce the equation to the form $x^2+px=q$.

Complete the square, by adding to both members the square of one-half the coefficient of x.

Equate the square root of the first member to the $\pm$ square root of the second, and solve the linear equations thus formed.

230. 1. Solve the equation $3x^2-8x=-4$.

Dividing by 3, $\quad x^2-\dfrac{8x}{3}=-\dfrac{4}{3}$,

which is in the form $\quad x^2+px=q$.

Adding to both members the square of $\dfrac{4}{3}$, we have

$$x^2-\frac{8x}{3}+\left(\frac{4}{3}\right)^2=-\frac{4}{3}+\frac{16}{9}=\frac{4}{9}.$$

Equating the square root of the first member to the $\pm$ square root of $\dfrac{4}{9}$, $\quad x-\dfrac{4}{3}=\pm\dfrac{2}{3}$.

Transposing $-\dfrac{4}{3}$, $\quad x=\dfrac{4}{3}\pm\dfrac{2}{3}=2$ or $\dfrac{2}{3}$.

If the *coefficient of x^2 is negative*, the *sign* of *each term* must be changed.

2. Solve the equation $-9x^2-21x=10$.

Dividing by -9, $\quad x^2+\dfrac{7x}{3}=-\dfrac{10}{9}$.

Adding to both members the square of $\frac{7}{6}$,

$$x^2+\frac{7x}{3}+\left(\frac{7}{6}\right)^2=-\frac{10}{9}+\frac{49}{36}=\frac{9}{36}.$$

Extracting square roots, $\qquad x+\frac{7}{6}=\pm\frac{3}{6}.$

Then, $\qquad x=-\frac{7}{6}\pm\frac{3}{6}=-\frac{2}{3}$ or $-\frac{5}{3}.$

EXERCISE 100

Solve the following equations and verify each result:

1. $t^2+4t=32.$
2. $u^2-u=6.$
3. $v^2-8v=-12.$
4. $m^2-2m=15.$
5. $4x^2+4x=3.$
6. $12k^2-k=1.$
7. $9t^2-3t=2.$
8. $9t^2-9t=-2.$
9. $9t^2+9t=4.$
10. $16z^2-8z=15.$

231. The graphs of affected quadratic equations can be readily constructed by the method used in § 227.

Construct the graph or geometrical picture of

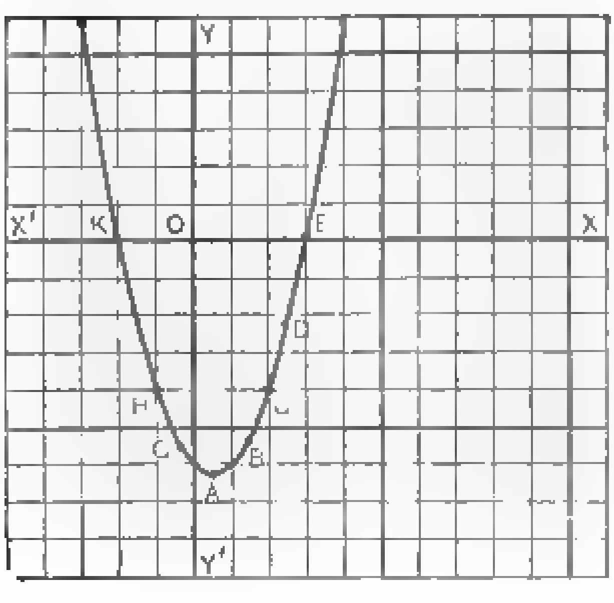
$$x^2-x-6=0. \tag{1}$$

Placing the first member of the equation equal to y, we have

$$x^2-x-6=y. \tag{2}$$

Assigning values to x, we obtain corresponding values of y. For example,

Substituting $x=0$ in (2), we have $y=-6$,
Substituting $x=2$ in (2), we have $y=-4$, etc.



$x^2-x-6=y$

x	y
0	-6
$\frac12$	$-6\frac14$ (A)
1	-6
$\frac32$	$-5\frac14$ (B)
2	-4 (C)
$\frac52$	$-2\frac14$ (D)
3	0 (E)
4	6
$-\frac12$	$-5\frac14$ (G)
-1	-4 (H)
-2	0 (K)
-3	6

Solving $x^2-x-6=0$

or $(x-3)(x+2)=0,$

we have, $x=3$ or $-2.$

In the graph of this affected quadratic equation note

(*a*) that the lowest point of the curve is not on the y-axis;

(*b*) that the curve crosses the x-axis in two points ($x=3$, $x=-2$) corresponding to the algebraic solution.

The graph of every equation of the form $x^2+px=q$ or $ax^2+bx+c=0$ is a curve of the above form and is called a *parabola.*

EXERCISE 101

Construct the graph of the following equations and compare the points of intersection with the algebraic solution:

1. $x^2-x-2=0.$
2. $x^2-8x+15=0.$
3. $x^2+6x=-8.$
4. $8x^2+6x=-1.$
5. $8x^2-2x=1.$
6. $3x^2-17x-6=0.$

232. Second Method of Completing the Square.

Every affected quadratic equation can be reduced to the form $ax^2+bx+c=0$, or $ax^2+bx=-c$.

Multiplying both members by $4a$, we have

$$4a^2x^2+4abx=-4ac.$$

We complete the square by adding to both members the square of $\frac{4ab}{2\times 2a}$ or b. (If the coefficient of x^2 is a perfect square, the trinomial square may be completed by adding *to both members the square of the quotient obtained by dividing the coefficient of x by twice the square root of the coefficient of x^2.* § 229.)

Then, $$4a^2x^2+4abx+b^2=b^2-4ac.$$

Extracting square roots, $2ax+b=\pm\sqrt{b^2-4ac}.$

Transposing, $$2ax=-b\pm\sqrt{b^2-4ac}.$$

Whence, $$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

RULE: — Reduce the equation to the form $ax^2+bx=-c$. Multiply both members by four times the coefficient of x^2, and add to each the square of the coefficient of x in the given equation.

The only advantage of this method over the preceding is in avoiding fractions in completing the square.

1. Solve the equation $2x^2-7x=-3$.

Multiplying both members by 4×2, or 8,

$$16x^2-56x=-24.$$

Adding to both members the square of 7,

$$16x^2-56x+7^2=-24+49=25.$$

Extracting square roots, $4x-7=\pm 5$.

Then, $4x=7\pm 5=12$ or 2, and $x=3$ or $\frac{1}{2}$.

EXERCISE 102

Solve the following equations using the second method; verify all results:

1. $3m^2+10m=-3$.
2. $6t^2-13t=-6$.
3. $2r^2-15r+25=0$.
4. $5u^2+3u-2=0$.
5. $4x^2+2x-1=0$.
6. $15m^2+16m+1=0$.
7. $12x^2-11x=-2$.
8. $6x^2+11x=7$.
9. $6x^2-7x=20$.
10. $10q^2+3q=1$.

233. Solution of Affected Quadratic Equations by Formula.

It follows from § 232 that, if $ax^2+bx+c=0$,

then
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}. \qquad (1)$$

This result may be used as a *formula* for the solution of any affected quadratic equation in the form $ax^2+bx+c=0$.

1. Solve the equation $2x^2+5x-18=0$.

Here, $a=2$, $b=5$, and $c=-18$; substituting in (1),

$$x=\frac{-5\pm\sqrt{25+144}}{4}=\frac{-5\pm 13}{4}=2 \text{ or } -\frac{9}{2}.$$

2. Solve the equation $-5x^2+14x+3=0$.

Here, $a=-5, b=14, c=3$; substituting in (1),

$$x=\frac{-14\pm\sqrt{196+60}}{-10}=\frac{-14\pm16}{-10}=-\frac{1}{5}\text{ or }3.$$

3. Solve the equation $110x^2-21x=-1$.

Here, $a=110$, $b=-21$, $c=1$; then,

$$x=\frac{21\pm\sqrt{441-440}}{220}=\frac{21\pm1}{220}=\frac{1}{10}\text{ or }\frac{1}{11}.$$

Particular attention must be paid to the *signs* of the coefficients in making the substitution.

EXERCISE 103

Solve the following equations by formula:

1. $4x^2-7x=-3$.
2. $9u^2+22u=-8$.
3. $8t^2+10t=3$.
4. $3v^2-8v-3=0$.
5. $12=23e-5e^2$
6. $8x^2+2x=3$.
7. $3t^2-2t=40$.
8. $m^2+7m=18$.
9. $28x^2-x-15=0$.
10. $5x^2-17x+6=0$.

234. The formula in § 233 is important in determining the nature of the roots of a quadratic equation, also in determining the relation between the roots and the coefficients in the equation.

In $ax^2+bx+c=0$,

by § 233, $$x=\frac{-b+\sqrt{b^2-4ac}}{2a}\text{ or }\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

Call the first root r_1 and the second r_2.

I. If b^2-4ac is *positive*,

r_1 and r_2 are *real* and *unequal*.

Ex., $x^2-2x-8=0$, $b^2-4ac=4+32=+$.

Solving, $x=4$ or -2.

See Figure 1, Plate III.

II. If $b^2-4ac=0$,

r_1 and r_2 are *real* and *equal*.

Ex., $x^2-2x+1=0$, $b^2-4ac=4-4=0$.

Solving, $x=1$ or 1.

See Figure 2, Plate III.

III. If b^2-4ac is *negative*,

r_1 and r_2 are *imaginary* (§ 218).

Ex., $x^2-2x+3=0$, $b^2-4ac=4-12$.

Solving, $$x=\frac{2\pm\sqrt{4-12}}{2}=1+\sqrt{-2} \text{ or } 1-\sqrt{-2}.$$

See Figure 3, Plate III.

The *intersection* of the curve with the x-axis is *imaginary*.

Imaginary roots always occur in *conjugate pairs* (§ 214).

Note that these three equations differ only in the third terms and that this difference seems to have the effect of raising or lowering the curve with respect to the x-axis.

Adding the values of r_1 and r_2, in

$$r_1=\frac{-b+\sqrt{b^2-4ac}}{2a},\quad r_2=\frac{-b-\sqrt{b^2-4ac}}{2a},$$

we have $$r_1+r_2=\frac{-2b}{2a}=\frac{-b}{a}.$$

Finding their product,

$$r_1r_2=\frac{b^2-(b^2-4ac)}{2a}\ (\S\ 89)=\frac{4ac}{4a^2}=\frac{c}{a}.$$

Hence, if a quadratic equation is in the form

$$ax^2+bx+c=0,$$

the sum of the roots equals minus the coefficient of x divided by the coefficient of x^2, and the product of the roots equals the independent term divided by the coefficient of x^2.

1. Find by inspection the sum and product of the roots of

$$3x^2-7x-15=0.$$

The sum of the roots is $\frac{7}{3}$, and their product $\frac{-15}{3}$, or -5.

2. One root of the equation $6x^2+31x=-35$ is $-\frac{7}{2}$; find the other.

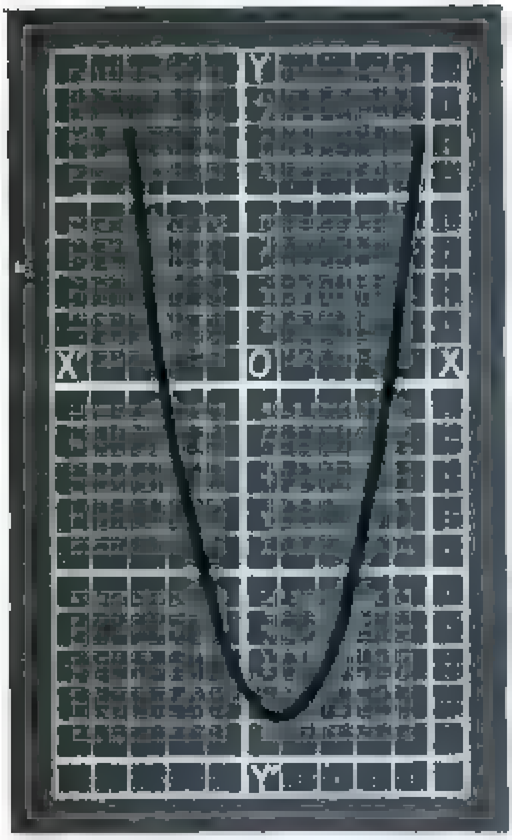


Fig. 1. $x^2-2x-8=0$

Fig. 1

x	y
0	-8
1	-9
2	8
3	-5
4	0
5	7
-1	5
-2	0
-3	7

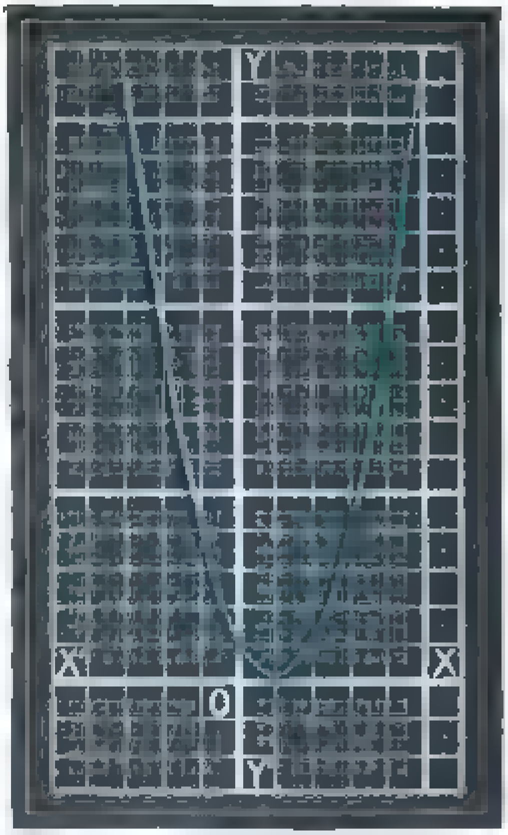


Fig. 2. $x^2-2x+1=0$

Fig. 2

x	y
0	1
1	0
2	1
3	4
4	9
5	16
-2	9
-3	16

Fig. 3

x	y
0	3
1	2
2	3
3	6
4	11
-1	6
-2	11
-3	18

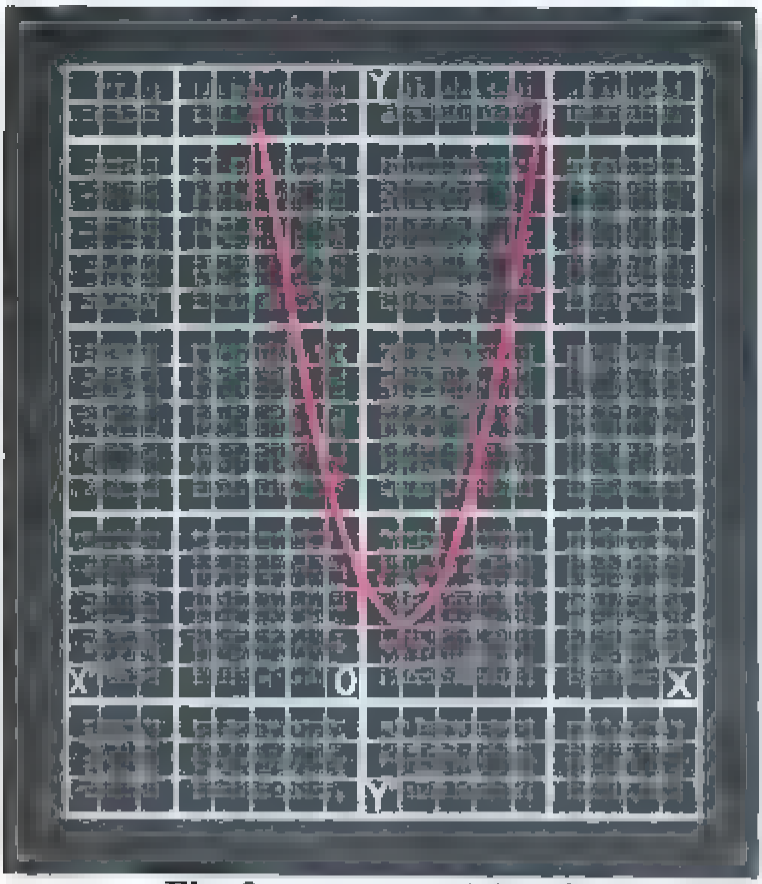


Fig. 3. $x^2-2x+3=0$

PLATE III

The equation can be written $6x^2+31x+35=0$.

Then, the sum of the roots is $-\frac{31}{6}$.

Hence, the other root is $-\frac{31}{6}-\left(-\frac{7}{2}\right)$, or $-\frac{31}{6}+\frac{7}{2}$, or $-\frac{5}{3}$.

We may also find the other root by dividing the *product* of the roots, $\frac{35}{6}$, by $-\frac{7}{2}$.

We may find the values of certain other expressions which are symmetrical in the roots of the quadratic.

3. If r_1 and r_2 are the roots of $ax^2+bx+c=0$, find the value of $r_1^2+r_1r_2+r_2^2$.

We have, $\quad r_1^2+r_1r_2+r_2^2=(r_1+r_2)^2-r_1r_2$.

But, $\quad r_1+r_2=-\frac{b}{a}$, and $r_1r_2=\frac{c}{a}$.

Whence, $\quad r_1^2+r_1r_2+r_2^2=\frac{b^2}{a^2}-\frac{c}{a}=\frac{b^2-ac}{a^2}$.

4. Determine by inspection the nature of the roots of

$$2x^2-5x-18=0.$$

Here $a=2$, $b=-5$, $c=-18$; and $b^2-4ac=25+144=169$.
Since b^2-4ac is positive, the roots are real and unequal.
Since b^2-4ac is a perfect square, both roots are rational.

EXERCISE 104

Find by inspection the nature of the roots, the sum and product of the roots, and *construct* the *graph* of each of the following 8 problems:

1. $x^2+8x+7=0$.
2. $x^2-x-20=0$.
3. $4x^2-x-5=0$.
4. $6x^2+x=0$.
5. $x^2+2x+4=0$.
6. $9x^2+6x-1=0$.
7. $9x^2+6x+1=0$.
8. $25x^2-4=0$.
9. One root of $x^2+7x=98$ is 7; find the other.

Note that your definitions §§ 39, 60 are involved in these examples.

10. One root of $5x^2-17x+6=0$ is $\frac{2}{5}$; find the other.
11. Is 5 a root of $x^2+5x+5=0$?

If r_1 and r_2 are the roots of $ax^2+bx+c=0$, find the values of:

12. $\frac{r_1^2+r_2^2}{r_1 r_2}$. 13. $\frac{1}{r_1}+\frac{1}{r_2}$. 14. $\frac{1}{r_1^2}+\frac{1}{r_2^2}$.

15. $r_1^3+r_2^3$. [Hint: $(x+y)^3+(x-y)^3$ contains but two terms.]

EXERCISE 105

Solve the following equations by the method which seems best adapted to the example under consideration, verifying each result:

(In solving any equation, we reject any solution which does not satisfy the given equation.)

1. $4t^2+3t=10$.
2. $49x^2+49x+10=0$.
3. $5h^2+12h=-4$.
4. $32v-48v^2=-3$.
5. $9m^2+6m=19$.
6. $2r^2-15r=-13$.
7. $12x^2+5x+1=0$.
8. $10-21k-10k^2=0$.
9. $\frac{2t+3}{8+t}-\frac{2t+9}{3t+4}=0$.
10. $4y+\frac{14-y}{y+1}=14$.
11. $\frac{z}{5-z}-\frac{5-z}{z}=\frac{15}{4}$.
12. $\sqrt{3+x-x^2}=2x-3$.
13. $\sqrt[3]{5s+11}=\sqrt{3s+1}+2$.
14. $\frac{x-2}{x+5}-\frac{x+4}{x-3}=-\frac{7}{3}$.

(Compare Ex. 19, Exercise 56.)

15. $\frac{x+1}{x+2}-\frac{x+3}{x+4}=\frac{8}{3}$.
16. $\frac{2x^2+3x-5}{2x^2-x-1}=\frac{3x^2+4x-1}{3x^2-2x+7}$.
17. $\frac{2t^2-4t-3}{2t^2-2t+3}=\frac{t^2-4t+2}{t^2-3t+2}$.
18. $\frac{1}{m^2-4}-\frac{1}{3(m+2)}=1+\frac{3}{2-m}$.
19. $\sqrt{8y+7}=\sqrt{4y+3}+\sqrt{2y+2}$.
20. $\frac{x+1}{x-1}+\frac{x+2}{x-2}+\frac{x+3}{x-3}=3$.

(Compare Ex. 14.)

21. $(x-2)(x+3)(x-4)=0.$

22. $(x-3)(2x^2+13x+20)=0.$

23. $x^2+x+1=0.$

24. $x^3=1.$

25. $\frac{1}{1-t^2}+\frac{1}{1+t}-\frac{1}{1-t}=-\frac{7}{8}.$

26. $3-\frac{1}{x+2}-\frac{3}{2(2x-3)}=\frac{5}{(x+2)(2x-3)}.$

27. $\frac{x+1}{x-1}+\frac{x+2}{x-2}=\frac{2x+13}{x+1}.$

28. $t^3=-8.$ (The roots are the three *different* cube roots of -8. Compare Ex. 24.)

29. $\frac{\sqrt{v}}{\sqrt{v+2}}-\frac{\sqrt{v+2}}{\sqrt{v}}=\frac{5}{6}.$

30. $\frac{2}{m-2}+\frac{2}{m-5}=\frac{m^2+3m-16}{m^2-7m+10}.$

31. $x^2+ax-bx-ab=0.$

We may write the equation $x^2+(a-b)x=ab.$

Multiplying both members by 4 times the coefficient of x^2,

$$4x^2+4(a-b)x=4ab.$$

Adding to both members the square of $a-b$,

$$4x^2+4(a-b)x+(a-b)^2=4ab+a^2-2ab+b^2$$
$$=a^2+2ab+b^2.$$

Extracting square root, $2x+(a-b)=\pm(a+b).$

Or, $2x=-(a-b)\pm(a+b).$

Then, $2x=-a+b+a+b=2b,$

or $2x=-a+b-a-b=-2a.$

Whence, $x=b$ or $-a.$

If several terms contain the same power of x, the coefficient of that power should be written in parenthesis, as shown in Ex. 1.

For the solution of literal affected quadratic equations, the methods of § 232 are usually most convenient.

The above equation can be solved more easily by the method of § 103; thus, by § 101, the equation may be written

$$(x+a)(x-b)=0.$$

Then, $x+a=0$, or $x=-a$;

and $x-b=0$, or $x=b$.

Several equations in Exercise 105 may be solved most easily by the method of § 103.

32. Solve the equation $(m-1)x^2-2m^2x=-4m^2$.

Multiplying both members by $m-1$, and adding to both the square of m^2,
$$(m-1)^2x^2-2m^2(m-1)x+m^4=-4m^2(m-1)+m^4$$
$$=m^4-4m^3+4m^2.$$

Extracting square root, $(m-1)x-m^2=\pm(m^2-2m)$.

Then, $(m-1)x=m^2+m^2-2m$ or m^2-m^2+2m
$$=2m(m-1) \text{ or } 2m$$

Whence, $x=2m$ or $\dfrac{2m}{m-1}$.

33. $x^2-mx=m^2$; solve for x.

34. $x^2-mx=m^2$; solve for m.

Solve the following for x:

35. $x^2-2ax=-6a+9$.

36. $x^2-(a-b)x=ab$.

37. $x^2+nx+x=-n$.

38. $x^2-m^2kx+mk^2x=m^3k^3$.

39. $\sqrt{a+x}-\sqrt{2x}=\dfrac{2a}{\sqrt{a+x}}$.

40. $(a+b)x^2+(3a+b)x=-2a$.

41. $\sqrt{x-a}+\sqrt{2x+3a}=5a$.

Solve for t:

42. $\sqrt{5a+t}+\sqrt{5a-t}=2\sqrt{t}$.

43. $\sqrt{t^2-\sqrt{2t+1}}=t-1$.

44. $\sqrt{t+9a}+\sqrt{25a-t}=\sqrt{2t+32a}$.

45. $S=V_0t+\frac{1}{2}gt^2$.

46. $\dfrac{1+a}{1-at}+\dfrac{1-a}{1+at}=1$.

47. $\dfrac{t+a}{t+b}+\dfrac{t+b}{t+a}=\dfrac{5}{2}$.
(Compare Ex. 14.)

48. Solve for g: $t=\pi\sqrt{\dfrac{l}{g}}$.

49. Solve for s: $V=\sqrt{2gs}$.

50. Solve for n: $\dfrac{2n-3a}{3n+a}+\dfrac{3n+a}{2n-3a}=\dfrac{10}{3}$.

51. Solve for n: $S=\dfrac{n}{2}[2a+(n-1)d]$.

PROBLEMS INVOLVING QUADRATIC EQUATIONS WITH ONE UNKNOWN NUMBER

235. In solving problems which involve quadratic equations, there will usually be two values of the unknown number; only those values should be retained which satisfy the conditions of the problem.

1. A man sold a watch for \$21, and lost as many per cent as the watch cost dollars. Find the cost of the watch.

Let x = number of dollars the watch cost.

Then, x = the per cent of loss,

and $x \times \frac{x}{100}$, or $\frac{x^2}{100}$ = number of dollars lost.

By the conditions, $$\frac{x^2}{100} = x - 21.$$

Solving, $$x = 30 \text{ or } 70.$$

Then, the cost of the watch was either \$30 or \$70; for either of these answers satisfies the conditions of the problem.

2. A farmer bought some sheep for \$72. If he had bought 6 more for the same money, they would have cost him \$1 apiece less. How many did he buy?

Let n = number bought.

Then, $\frac{72}{n}$ = number of dollars paid for one,

and $\frac{72}{n+6}$ = number of dollars paid for one if there had been 6 more.

By the conditions, $$\frac{72}{n} = \frac{72}{n+6} + 1.$$

Solving, $$n = 18 \text{ or } -24.$$

Only the *positive* value is admissible, for the negative value does not satisfy the conditions of the problem.

Therefore, the number of sheep was 18.

If, in the enunciation of the problem, the words "6 more" had been changed to "6 fewer," and "\$1 apiece less" to "\$1 apiece more," we should have found the answer 24.

3. If 3 times the square of the number of trees in an orchard be increased by 14, the result equals 23 times the number; find the number.

Let x = number of trees.

By the conditions, $3x^2 + 14 = 23x$.

Solving, $x = 7$ or $\frac{2}{3}$.

Only the first value of x is admissible, for the fractional value does not satisfy the conditions of the problem.

Then, the number of trees is 7.

4. If the square of the number of dollars in a man's assets equals 5 times the number increased by 150, find the number.

Let x = number of dollars in his assets.

By the conditions, $x^2 = 5x + 150$.

Solving, $x = 15$ or -10.

This means that he has assets of \$15, or liabilities of \$10.

EXERCISE 106

Verify all results.

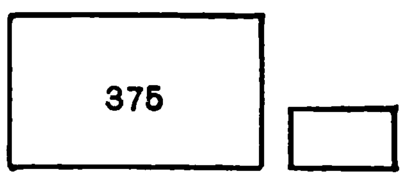
1. What number added to its reciprocal gives $\frac{25}{12}$?

2. Divide 17 into two such parts that three times the square of the greater shall exceed twice the square of the less by 115.

3. Find three consecutive numbers such that if the square of the second number be subtracted from the sum of the squares of the first and third, the remainder will be 38.

4. The sum of two numbers is 3 and the sum of their cubes is 7; find the numbers.

5. Two rectangles have their corresponding sides in the ratio of 5 to 2. In the greater the ratio of the length to the breadth is $\frac{5}{3}$. The area of the greater is 375; find the area of the less.



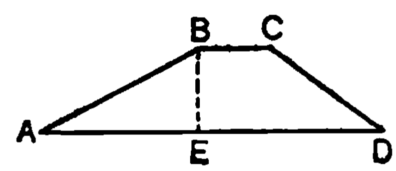
6. A farmer bought a certain number of sheep for \$300. Having lost 7, he sold the rest for \$2 a head more than they cost him, and gained \$44. How many did he sell?

7. A rectangular field is twice as long as it is wide. If 20 rods were subtracted from the length and the same amount were added to the width, the field would be square and would contain $22\frac{1}{2}$ acres. Would this change decrease or increase the area of the field?

8. A fast train's schedule from New York to Chicago is 12 miles an hour faster than a slow one, and requires 5 less hours to travel 960 miles. Find the rate of each train.

9. If the product of three consecutive numbers be divided by each of them in turn, the sum of the quotients is 107; find the numbers.

10. The area of a trapezoid is equal to the product of one-half the sum of the parallel sides and the altitude. Find the sides and altitude of trapezoid $ABCD$ in which AD is 8 feet more than BC, and EB 2 feet less than BC, the area being 55 square feet. Are there two such trapezoids?



11. A merchant sold a bill of goods for \$24, making as many per cent as the goods cost dollars. Find the cost.

12. Find two numbers whose difference is 4, and the difference of whose cubes is 3088.

13. The area of a certain square field exceeds that of another square field by 1008 square yards, and the perimeter of the greater exceeds one-half that of the smaller by 120 yards. Find the side of each field.

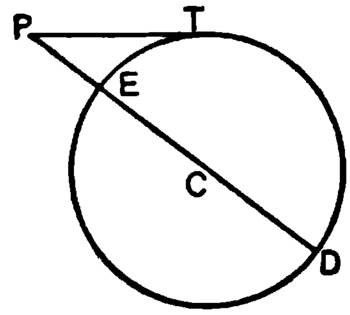
14. A and B set out at the same time from places 247 miles apart, and travel toward each other. A's rate is 9 miles an hour; and B's rate in miles an hour is less by 3 than the number of hours at the end of which they meet. Find B's rate.

15. A man buys a certain number of shares of stock, paying for each as many dollars as he buys shares. After the price has advanced one-fifth as many dollars per share as he has shares, he sells, and gains \$980. How many shares did he buy?

16. The two digits of a number differ by 1; and if the square of the number be added to the square of the given number with its digits reversed, the sum is 585. Find the number.

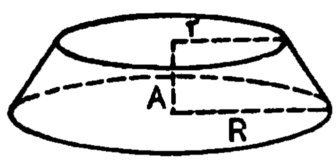
17. A merchant sold two pieces of cloth of different quality for \$105, the poorer containing 28 yards. He received for the finer as many dollars a yard as there were yards in the piece; and 7 yards of the poorer sold for as much as 2 yards of the finer. Find the value of each piece.

18. In a circle with centre at C, the tangent PT is a mean proportional between the whole secant PD and the external part PE. If the tangent is 8, and the diameter ED is 12, find PE.

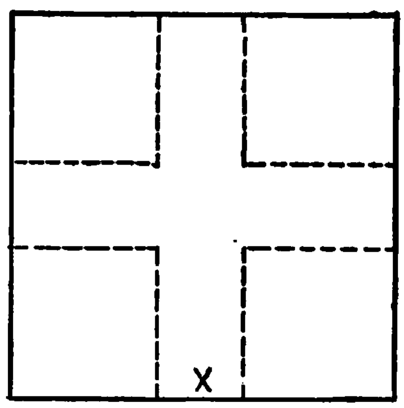


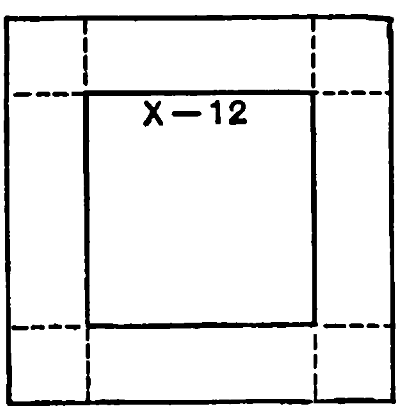
19. A and B gained in trade \$2100. A's money was in the firm 15 months, and he received in principal and gain \$3900. B's money, which was \$5000, was in the firm 12 months. How much money did A put into the firm?

20. The formula for the volume of the frustum of a cone is $V=\frac{1}{3}\pi A(R^2+r^2+Rr)$, in which r is the radius of the upper base, R the radius of the lower base, A the altitude and V the volume. If $V=872\pi$, $r=10$ and $A=6$; find R.



21. A square garden plot containing 144 square feet has two walks of equal width intersecting at right angles to each other and to the sides of the garden. The area of the walk is one-half the area of the entire square; find the width of the walk.

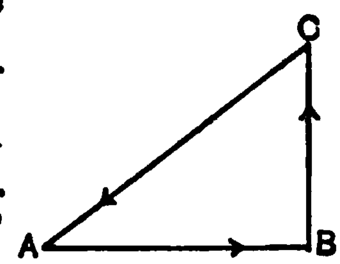




22. A square piece of tin is to be made into a rectangular box by cutting a square out of each corner and folding up the sides. The pieces cut out are 6 inches square; the volume of the box, 1944 cubic inches. How large was the sheet of tin? If a cubical box had been cut from this

sheet of tin, would its volume have been greater or less than that of the first box formed?

23. In a right-angled triangle, ABC, one side is 5 more than the other, and the hypotenuse is 5 more than the longer side. Find the dimensions. Draw diagram explaining your solutions.



24. If a body is thrown downward with an initial velocity, v_0, then the space it passes over in t seconds is found to be given by the equation

$$S=v_0t+\frac{1}{2}gt^2.$$

A stone was thrown downward with a velocity of 40 feet per second from a balloon a mile high; g is 32.15. How many seconds elapsed before the stone reached the earth?

25. In the equation $F=\frac{mM}{d^2}$, M and m represent the masses of any two attracting bodies, as, for instance, the earth and the moon, d represents the distance between these bodies, and F the force with which they attract each other.

If the moon had twice its present mass and were twice as far from the earth as at present, how much greater or less would the force of the earth's attraction be upon it than at present?

26. In the equation $E=\frac{1}{2}mv^2$, E represents the energy of a moving body, the mass of which is m and the velocity is v. Compare the energies of two bodies, one of which has twice the mass and twice the velocity of the other.

27. When a bullet is shot upward with a velocity, v, the height, S, to which it rises is given by the equation

$$v=\sqrt{2gS}.$$

Find with what velocity a body must be thrown upward to rise to the height of the Washington monument (555 feet). (See Problem 25, Exercise 99.)

236. Equations in Quadratic Form.

An equation is said to be in the *quadratic form* when it is expressed in three terms, two of which contain the unknown number, and *the exponent of the unknown number in one of these terms is twice its exponent in the other;* as,

$$x^6 - 6x^3 = 16; \quad x^3 + x^{\frac{3}{2}} - 72 = 0; \text{ etc.}$$

In equations in quadratic form, the simplest method for the beginner to apply is to let some letter represent the lowest power of the unknown quantity in the given equation.

1. $x^6 - 6x^3 = 16$. Let $y = x^3$.

Then, $y^2 - 6y - 16 = 0$.

Whence, $y = 8$ or -2,

$x^3 = 8$ or -2,

$x = 2$ or $-\sqrt[3]{2}$.

Verify these roots.

2. $2x + 3\sqrt{x} = 27$. Let $y = x^{\frac{1}{2}}$ or $\sqrt{x}$.

Then, $2y^2 + 3y = 27$,

$(2y + 9)(y - 3) = 0$,

$y = 3$ or $-\frac{9}{2}$,

$\sqrt{x} = 3$ or $-\frac{9}{2}$,

$x = 9$ or $\frac{81}{4}$.

Verify these results.

3. $2s^{-8} - 35s^{-4} + 48 = 0$. Let $x = s^{-4}$.

Then, $2x^2 - 35x + 48 = 0$.

Whence, $x = 16$ or $\frac{3}{2}$,

$s^{-4} = 16$ or $\frac{3}{2}$,

$\frac{1}{s^4} = 16$ or $\frac{3}{2}$,

$s^4 = \frac{1}{16}$ or $\frac{2}{3}$,

$s = \pm\frac{1}{2}$ or $\pm\sqrt[4]{\frac{2}{3}}$

EXERCISE 107

1. $z^4-29z^2=-100.$
2. $y^{-6}+19y^{-3}=216.$
3. $8t+14\sqrt{t}=15.$
4. $m^3-3m^{\frac{3}{2}}=88.$
5. $2x^{-5}+61x^{-\frac{5}{2}}-96=0.$
6. $32x^5+\dfrac{1}{x^5}=-33.$
7. $6m^{-\frac{2}{3}}-5m^{-\frac{1}{3}}=6.$
8. $4\sqrt[5]{x^4}+6=11\sqrt[5]{x^2}.$
9. $(2x^2-3x)^2-8(2x^2-3x)=9.$
10. $(5m+12)-5(5m+12)^{\frac{1}{2}}=-4.$

FACTORING

237. Factoring of Quadratic Expressions.

A *quadratic expression* is an expression of the form

$$ax^2+bx+c.$$

In § 94 we showed how to factor certain expressions of this form *by inspection;* we will now derive a rule for factoring any quadratic expression; we have,

$$\begin{aligned} ax^2+bx+c &= a\left(x^2+\frac{bx}{a}+\frac{c}{a}\right) \\ &= a\left[x^2+\frac{bx}{a}+\left(\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}\right] \\ &= a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right] \\ &= a\left(x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\right)\left(x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\right), \end{aligned}$$

by § 89.

But by § 233, the roots of $ax^2+bx+c=0$ are

$$-\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a} \text{ and } -\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}.$$

Hence, to factor a quadratic expression, place it equal to zero, and solve the equation thus formed.

Then the required factors are the coefficient of x^2 in the given expression, x minus the first root, and x minus the second.

1. Factor $6x^2+7x-3$.

Solving the equation $6x^2+7x-3=0$, by § 233,

$$x=\frac{-7\pm\sqrt{49+72}}{12}=\frac{-7\pm 11}{12}=\frac{1}{3} \text{ or } -\frac{3}{2}.$$

Then, $$6x^2+7x-3=6\left(x-\frac{1}{3}\right)\left(x+\frac{3}{2}\right)$$

$$=3\left(x-\frac{1}{3}\right)\times 2\left(x+\frac{3}{2}\right)=(3x-1)(2x+3).$$

2. Factor $4+13x-12x^2$.

Solving the equation $4+13x-12x^2=0$, by § 233,

$$\therefore x=\frac{-13\pm\sqrt{169+192}}{-24}=\frac{-13\pm 19}{-24}=-\frac{1}{4} \text{ or } \frac{4}{3}.$$

Whence, $$4+13x-12x^2=-12\left(x+\frac{1}{4}\right)\left(x-\frac{4}{3}\right)$$

$$=4\left(x+\frac{1}{4}\right)\times\left(-3\right)\left(x-\frac{4}{3}\right)$$

$$=(1+4x)(4-3x).$$

3. Factor $2x^2-3xy-2y^2-7x+4y+6$.

We solve $$2x^2-x(3y+7)-2y^2+4y+6=0.$$

By § 233, $$x=\frac{3y+7\pm\sqrt{(3y+7)^2+16y^2-32y-48}}{4}$$

$$=\frac{3y+7\pm\sqrt{25y^2+10y+1}}{4}=\frac{3y+7\pm(5y+1)}{4}$$

$$=\frac{8y+8}{4} \text{ or } \frac{-2y+6}{4}=2y+2 \text{ or } \frac{-y+3}{2}.$$

Then, $$2x^2-3xy-2y^2-7x+4y+6$$

$$=2[x-(2y+2)]\left[x-\frac{-y+3}{2}\right]$$

$$=(x-2y-2)(2x+y-3).$$

EXERCISE 108

Factor the following:

1. $4x^2-12x-7$.
2. x^2+x-12.
3. $25x^2-10x-11$.
4. t^2+t+1.
5. $6t^2+3t+2$.
6. $36m^2-5m-1$.

7. $20x^2-13x+1$.
8. a^2+2a+2.
9. x^4+x.
10. $6-c-2c^2$.
11. $8v^2+18v-5$.
12. a^2+4a+1.
13. $a^2+ab-6b^2+a+13b-6$.
14. $2x^2-xy-y^2+3x+3y-2$.
15. $2x^2-4xy+x-6y^2+13y-6$.
16. $6a^2+7ab-4a-3b^2+5b-2$.

238. We will now take up the factoring of expressions of the forms $x^4+ax^2y^2+y^4$, or x^4+y^4, when the factors involve surds. (Compare § 96.)

1. Factor $a^4+2a^2b^2+25b^4$.

$$\begin{aligned}a^4+2a^2b^2+25b^4&=(a^4+10a^2b^2+25b^4)-8a^2b^2\\&=(a^2+5b^2)^2-(ab\sqrt{8})^2\\&=(a^2+5b^2+ab\sqrt{8})(a^2+5b^2-ab\sqrt{8})\\&=(a^2+2ab\sqrt{2}+5b^2)(a^2-2ab\sqrt{2}+5b^2).\end{aligned}$$

2. Factor x^4+1.

$$\begin{aligned}x^4+1&=(x^4+2x^2+1)-2x^2\\&=(x^2+1)^2-(x\sqrt{2})^2\\&=(x^2+x\sqrt{2}+1)(x^2-x\sqrt{2}+1).\end{aligned}$$

EXERCISE 109

In each of the following obtain two sets of factors, when this can be done without bringing in imaginary numbers:

1. x^4-7x^2+4.
2. a^4+b^4.
3. $9m^4-11m^2+1$.
4. $4a^4+6a^2+9$.
5. $36x^4-92x^2+49$.
6. $25m^4+28m^2n^2+16n^4$.

Solve the following:

7. $x^3+1=0$. (The three roots are the three different cube roots of -1.)
8. $x^4+2x^2+4=0$.
9. $x^4+8x=0$.
10. Find the three different cube roots of 27.

(Compare Ex. 24, Exercise 105.)

XV. SIMULTANEOUS QUADRATIC EQUATIONS

239. *On the use of the double signs $\pm$ and $\mp$.*

If two or more equations involve double signs, it will be understood that the equations can be read in two ways; first, reading all the *upper* signs together; second, reading all the *lower* signs together.

Thus, the equations $x = \pm 2$, $y = \pm 3$, can be read either

$$x = +2,\ y = +3, \text{ or } x = -2,\ y = -3.$$

Also, the equations $x = \pm 2$, $y = \mp 3$, can be read either

$$x = +2,\ y = -3, \text{ or } x = -2,\ y = +3.$$

240. Two equations of the second degree (§ 75) with two unknown numbers will generally produce, by elimination, an equation of the *fourth* degree with one unknown number.

Consider, for example, the equations

$$\begin{cases} x^2 + y = a. & (1) \\ x + y^2 = b. & (2) \end{cases}$$

From (1), $y = a - x^2$; substituting in (2),

$$x + a^2 - 2\,ax^2 + x^4 = b;$$

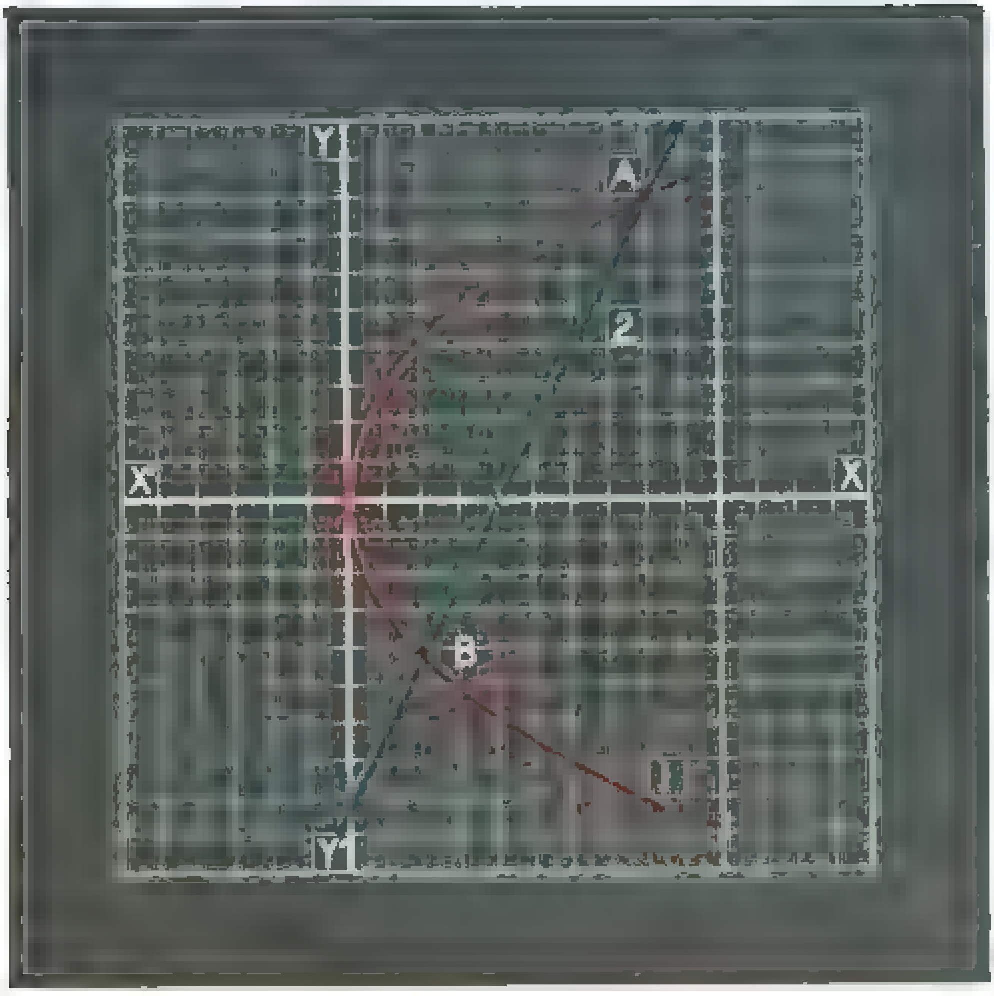
an equation of the fourth degree in x.

The methods already given are, therefore, not sufficient for the solution of every system of simultaneous quadratic equations, with two unknown numbers.

In certain cases, however, the solution may be effected. In the present work we shall consider only five simple types.

241. Type I. *When one equation is of the second degree, and the other of the first.*

Equations of this kind may be solved by finding one of the unknown numbers in terms of the other from the first degree equation, and substituting this value in the other equation.



(1)

$y^2 = 4x$

x	y
0	0
¼	1
1	2
2¼	3
4	4 (A)
¼	− 1
1	− 2 (B)
2¼	− 3
4	− 4

(2)

$y - 2x = -4$

x	y
0	− 4
1	− 2 (B)
2	0
3	2
4	4 (A)
− 1	− 6
− 2	− 8

The points A and B are the only points common to both curves Their coördinates, (4, 4) and (1, − 2), satisfy both equations and correspond to the two algebraic solutions.

In general there are two solutions of a quadratic equation and linear equation in two unknown quantities.

PLATE IV

Ex.
$$\begin{cases} y^2=4x. & (1) \\ y-2x=-4. & (2) \end{cases}$$

From (2) $\qquad y=2x-4. \qquad (3)$

Substituting in (1),
$$4x^2-16x+16=4x,$$
$$4x^2-20x+16=0,$$
$$x^2-5x+4=0,$$

whence, $\qquad x=4$ or $1.$

Substituting in (3),
$$\begin{aligned} y&=2x-4 \\ &=8-4, \text{ or } 2-4 \\ &=4, \text{ or } -2. \end{aligned}$$

The solution is $x=4$, $y=4$; or $x=1$, $y=-2$. Verify by substituting in the given equations. The graphs of these equations are given in Plate IV.

242. Type II. *When the given equations are symmetrical with respect to x and y; that is, when x and y can be interchanged without changing the equation.*

Equations of this kind may be solved by combining them in such a way as to obtain the values of $x+y$ and $x-y$.

Ex. Solve the equations
$$\begin{cases} x^2+y^2=50. & (1) \\ xy=-7. & (2) \end{cases}$$

Multiply (2) by 2, $\qquad 2xy=-14. \qquad (3)$

Add (1) and (3), $\qquad x^2+2xy+y^2=36$, or $x+y=\pm 6. \qquad (4)$

Subtract (3) from (1), $\qquad x^2-2xy+y^2=64$, or $x-y=\pm 8. \qquad (5)$

Add (4) and (5), $\qquad 2x=6\pm 8$, or -6 ± 8

Whence, $\qquad x=7, -1, 1$, or $-7.$

Subtract (5) from (4), $\qquad 2y=6\mp 8$, or $-6\mp 8.$

Whence, $\qquad y=-1, 7, -7$, or $1.$

The solution is $\quad x=\pm 7$, $y=\mp 1$; or, $x=\pm 1$, $y=\mp 7.$

Verify by substitution.

In subtracting ± 8 from 6, we have 6 ∓ 8, in accordance with the notation explained in § 239.

In operating with double signs, $\pm$ is changed to $\mp$, and $\mp$ to $\pm$, whenever $+$ should be changed to $-$.

The graphs of these equations will be found on Plate V. Note the symmetrical arrangement of the points of intersection.

243. Type III. *When one equation is of the third degree and the other is of the first degree.*

Certain forms of systems of first and third degree equations may be reduced to Type I or Type II by dividing one equation by the other.

Ex.
$$\begin{cases} x^3+y^3=18. & (1)\\ x+y=3. & (2)\end{cases}$$

Dividing (1) by (2), $x^2-xy+y^2=6.$ (3)

Use Type II, squaring (2) and subtracting the result from (3),

$$-3xy=-3.$$
$$-xy=-1. \qquad (4)$$

Adding (4) to (3), $x^2-2xy+y^2=5.$ (5)

$$x-y=\pm\sqrt{5}. \qquad (6)$$

Solving (6) and (2) by addition and subtraction:

$$x=\frac{3+\sqrt{5}}{2}, \text{ or } \frac{3-\sqrt{5}}{2},$$
$$y=\frac{3-\sqrt{5}}{2}, \text{ or } \frac{3+\sqrt{5}}{2}.$$

The solution is $x=\frac{3+\sqrt{5}}{2}$, $y=\frac{3-\sqrt{5}}{2}$, or

$$x=\frac{3-\sqrt{5}}{2}, \quad y=\frac{3+\sqrt{5}}{2}.$$

Verify by substitution in the given equations.

244. Type IV. *When each equation is in the form*

$$ax^2+by^2=c.$$

In this case, either x^2 or y^2 can be eliminated by addition or subtraction.

1. Solve the equations
$$\begin{cases} 3x^2+4y^2=76. & (1)\\ 3y^2-11x^2=4. & (2)\end{cases}$$

Multiply (1) by 3, $9x^2+12y^2=228.$

Multiply (2) by 4, $12y^2-44x^2=16.$

Subtracting, $53x^2=212.$

Then, $x^2=4$, and $x=\pm 2.$

Substituting $x=\pm 2$ in (1), $12+4y^2=76$, or $4y^2=64.$

Then, $y^2=16$, and $y=\pm 4.$

The solution is $x=2$, $y=\pm 4$; or, $x=-2$, $y=\pm 4.$

(1)

$$x^2+y^2=50$$

x	y
0	$\pm 5\sqrt{2}$
± 1	± 7 (A)
± 2	$\pm\sqrt{46}$
± 3	$\pm\sqrt{41}$
± 4	$\pm 6\sqrt{3}$
± 5	± 5
± 6	$\pm\sqrt{14}$
± 7	± 1 (C)

(2)

$$xy=-7$$

x	y
1	-7
2	$-7/2$
3	$-7/3$
4	$-7/4$
5	$-7/5$
6	$-7/6$
7	-1 (C)
-1	$+7$ (A)
-2	$+7/2$

etc.

In equation (1) since both x and y appear only in the second power, the double sign occurs in each substitution, so that for every pair of numerical values we obtain four points on the curve. E g. (± 1, ± 7) gives the four points A, B, C, D. The graph of equation (2) is in two branches. (See Ex. 4, § 245) In general two equations of the second degree in two unknowns give four solutions.

PLATE V

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In this case there are four possible sets of values of x and y which satisfy the given equations:

1. $x=2, y=4$.
2. $x=2, y=-4$.
3. $x=-2, y=4$.
4. $x=-2, y=-4$.

It would not be correct to leave the result in the form $x=\pm 2, y=\pm 4$, for this represents only the first and fourth of the above sets of values.

The method of elimination by addition or subtraction may be used in other examples.

2. Solve the equations $\begin{cases} 3x^2-4y=47. & (1) \\ 7x^2+6y=33. & (2) \end{cases}$

Multiply (1) by 3, $9x^2-12y=141.$

Multiply (2) by 2, $14x^2+12y=66.$

Adding, $23x^2=207.$

Then, $x^2=9$, and $x=\pm 3$.

Substituting $x=\pm 3$ in (1), $27-4y=47$, and $y=-5$.

It is possible to eliminate one unknown number, in the above examples, by *substitution* (§ 157), or by *comparison* (§ 158).

245. Type V. *When each equation is of the second degree, and homogeneous; that is, when each term involving the unknown numbers is of the second degree with respect to them* (§ 59).

Certain equations of this type can be solved by the methods of §§ 242 and 244. The method of Type V should be used only when the example cannot be solved by Type II or Type IV.

Ex. Solve $\begin{cases} x^2-2xy=5. & (1) \\ x^2+y^2=29. & (2) \end{cases}$

Dividing (1) by (2), $\dfrac{x^2-2xy}{x^2+y^2}=\dfrac{5}{29},$

or $29x^2-58xy=5x^2+5y^2.$

Then, $5y^2+58xy-24x^2=0$, or $(5y-2x)(y+12x)=0.$

Solving for y, $y=\dfrac{2x}{5}$, or $-12x$.

Substituting these values in (1) we have

$$x^2-\frac{4x^2}{5}=5, \quad \text{or} \quad x^2+24x^2=5.$$

Whence, $x=\pm 5$, or $x=\pm\frac{1}{\sqrt{5}}$.

$x=\pm 5$ was obtained through

$y=\frac{2x}{5}$, whence $y=\pm 2$.

$x=\pm\frac{1}{\sqrt{5}}$ was obtained through

$y=-12x$, whence $y=\mp\frac{12}{\sqrt{5}}$.

The solution is $x=\pm 5, y=\pm 2$, or $x=\pm\frac{1}{\sqrt{5}}, y=\mp\frac{12}{\sqrt{5}}$.

EXERCISE 110

1. $\begin{cases} 3x^2+2y^2=66. \\ 9x^2+5y^2=189. \end{cases}$

2. $\begin{cases} 3x-5y^2=-116. \\ 7x+4y^2=121. \end{cases}$

3. $\begin{cases} x^3+y^3=91. \\ x+y=7. \end{cases}$

4. $\begin{cases} x^2-xy+y^2=124. \\ x+y=8. \end{cases}$

5. $\begin{cases} 4t^2+u^2=61. \\ t^2+6u^2=159. \end{cases}$

6. $\begin{cases} x^3-y^3=-117. \\ x-y=-3. \end{cases}$

7. $\begin{cases} x+y=2. \\ xy=-15. \end{cases}$ (Type II.)

8. $\begin{cases} z^2+v^2=122. \\ z+v=-10. \end{cases}$

9. $\begin{cases} x^2+k^2=26. \\ kx=5. \end{cases}$

10. $\begin{cases} u-v=4. \\ 2uv=42. \end{cases}$

11. $\begin{cases} R^2+S^2=45. \\ R-S=3. \end{cases}$

12. $\begin{cases} R^3+S^3=9. \\ R+S=3. \end{cases}$

13. $\begin{cases} L^2+LM+M^2=19. \\ L-M=1. \end{cases}$

14. $\begin{cases} xy=25. \\ x+y=10. \end{cases}$

15. $\begin{cases} xy=24. \\ 2x-y=8. \end{cases}$

16. $\begin{cases} 9t^2-5u^2=205. \\ 4t^2+9u^2=136. \end{cases}$

17. $\begin{cases} 4h^2+7k^2=32. \\ 3h^2-11k^2=-41. \end{cases}$

18. $\begin{cases} \frac{2x}{3}+\frac{3y}{2}=2. \\ \frac{3}{2x}+\frac{2}{3y}=2. \end{cases}$

19. $\begin{cases} g^2+h^2=\frac{289}{36}. \\ gh=\frac{10}{3}. \end{cases}$

20. $\begin{cases} xy=ab. \\ x-y=a-b. \end{cases}$

21. From $v=gt$ and $S=\frac{1}{2}gt^2$, find v in terms of S and g.

22. From $C=\frac{E}{R}$ and $EC=\frac{H}{t}$, find H in terms of C, R, and t.

23. From $E=FS$, $F=ma$, $S=\frac{1}{2}at^2$, and $v=at$, find E in terms of m and v.

24. $\begin{cases} x^2+y^2=25. \\ x^2-xy=4. \end{cases}$

25. $\begin{cases} 5x^2-y^2=1. \\ xy-3y^2=-10. \end{cases}$

26. $\begin{cases} p^2+pq-5q^2=25. \\ p^2+4q^2=40. \end{cases}$

27. $\begin{cases} 2x^2-xy=28 \\ x^2+2y^2=18. \end{cases}$

GRAPHS

246. 1. Consider the equation $x^2+y^2=25$.

This means that for any point on the graph, the square of the abscissa, plus the square of the ordinate, equals 25.

But the square of the abscissa of any point, plus the square of the ordinate, equals the square of the distance of the point from the origin; for the distance is the hypotenuse of a right triangle, whose other two sides are the abscissa and ordinate. Then the square of the distance from O of any point on the graph is 25; or, the distance from O of any point on the graph is 5.

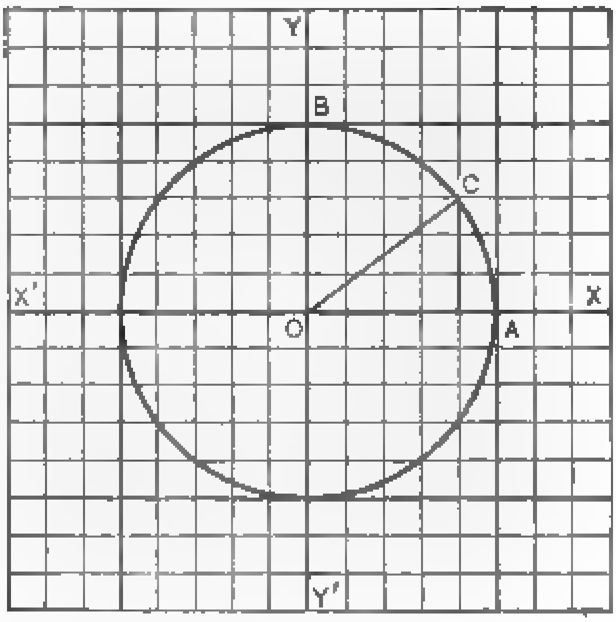


Fig. 1.

Thus, the graph is a circle of radius 5, having its centre at O.

(The graph of any equation of the form $x^2+y^2=a$ is a circle.) The graph of (1) Plate V is a circle.

2. Consider the equation $y^2=4x+4$.

If $x=0$,	$y^2=4$, or $y=\pm 2$.	(A, B)
If $x=1$,	$y^2=8$, or $y=\pm 2\sqrt{2}$.	(C, D)
If $x=-1$,	$y=0$. Etc.	(E)

The graph extends indefinitely to the right of YY'. (Fig. 2.)

If x is negative and <-1, y^2 is negative, and therefore y *imaginary*; then, no part of the graph lies to the left of E.

(The graph of Ex. 2 is a *parabola;* as also is the graph of any equation of the form $y^2=ax$ or $y^2=ax+b$. The graph of (1) § 241 is a parabola.)

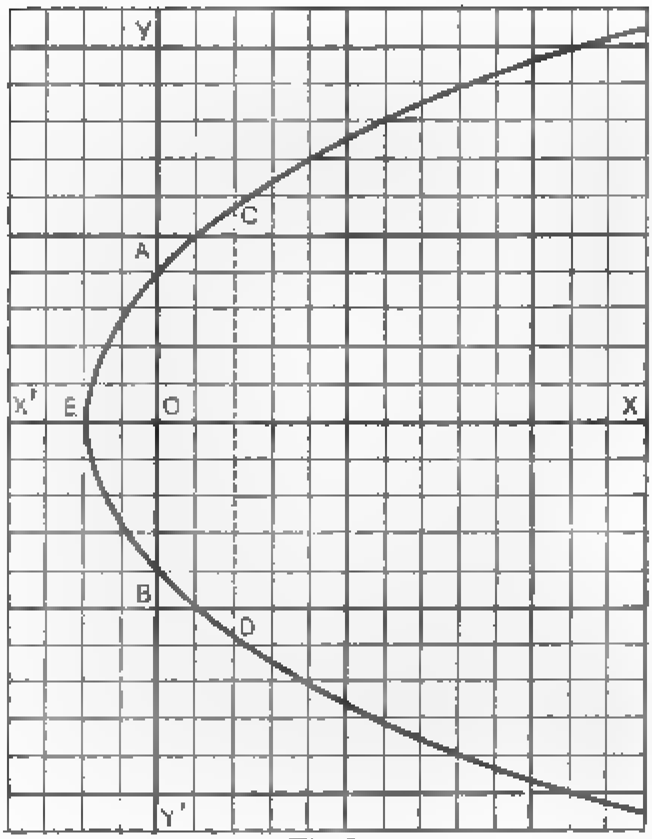


Fig. 2.

3. Consider the equation $x^2+4y^2=4$.

In this case it is convenient to first locate the points where the graph intersects the axes. (Fig. 3.)

If $y=0$, $x^2=4$,

or $x=\pm 2$. (A, A')

If $x=0$, $4y^2=4$,

or $y=\pm 1$. (B, B')

Putting $x=\pm 1$, $4y^2=3$,

$$y^2=\frac{3}{4}, \text{ or } y=\pm\frac{\sqrt{3}}{2}. \quad (C, D, C', D')$$

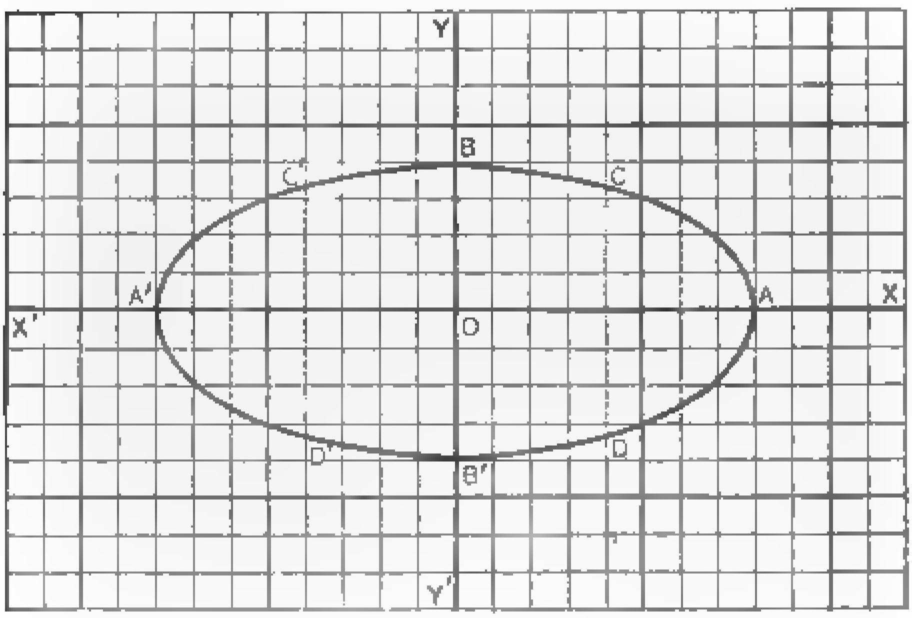


Fig. 3.

If x has any value >2, or <-2, y^2 is negative, and y imaginary; then, no part of the graph lies to the right of A, or left of A'.

If y has any value >1, or <-1, x^2 is negative, and x imaginary; then, no part of the graph lies above B, or below B'.

(The graph of Ex. 3 is an *ellipse*; as also is the graph of any equation of the form $ax^2+by^2=c$.)

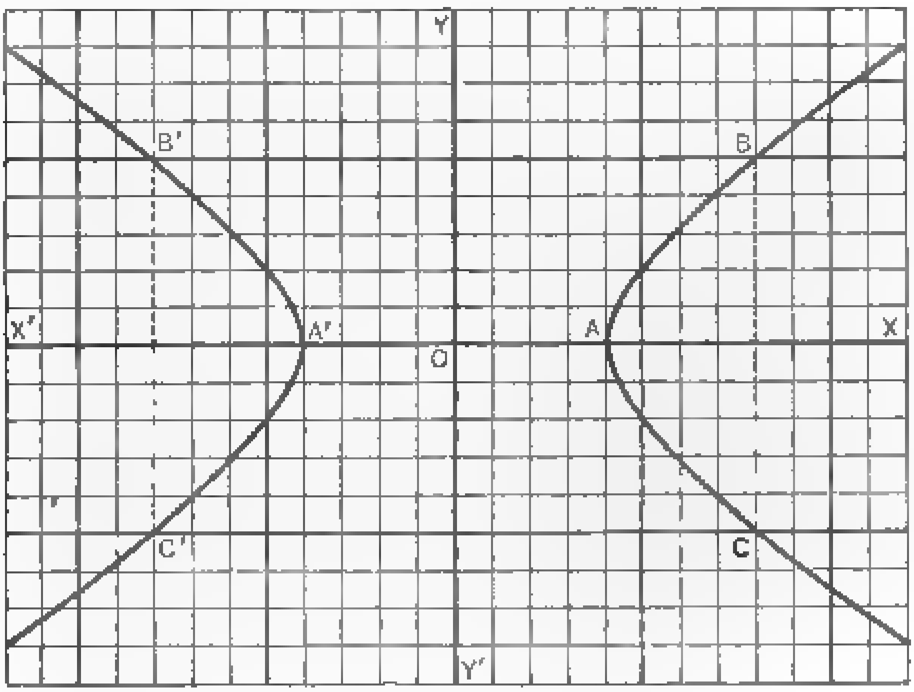


Fig 4.

4. Consider the equation $x^2-2y^2=1$.

Here $x^2-1=2y^2$, or $y^2=\dfrac{x^2-1}{2}$.

If $x=\pm 1$, $y^2=0$, or $y=0$. (A', A) (Fig. 4.)

If x has any value between 1 and -1, y^2 is negative, and y imaginary. Then, no part of the graph lies between A and A'.

If $x=\pm 2$, $y^2=\dfrac{3}{2}$, or $y=\pm\sqrt{\dfrac{3}{2}}$. (B, C, B', C')

The graph has two branches, BAC and $B'A'C'$, each of which extends to an indefinitely great distance from O.

(The graph of Ex. 4 is a *hyperbola*; as also is the graph of any equation of the form $ax^2-by^2=c$, or $xy=a$.) The graph of (2) Plate V is a hyperbola.

[redacted]

Find the graphs of the following sets of equations, and in each case verify the points of intersection by comparing with the algebraic solution:

1. $\begin{cases} x^2+4y^2=4. \\ x-y=1. \end{cases}$

2. $\begin{cases} x^2-4y=-7. \\ 2x+3y=4. \end{cases}$

3. $\begin{cases} x^2+y^2=29. \\ xy=10. \end{cases}$

4. $\begin{cases} x^2+y^2=4. \\ 3x-y=8. \end{cases}$

5. $\begin{cases} y^2-3x=-3. \\ x+2y=-2. \end{cases}$

6. $\begin{cases} x^2+y^2=13. \\ 4x-9y=6. \end{cases}$

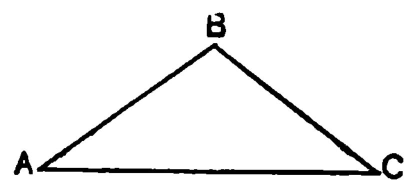
247. In solving problems which involve simultaneous equations of higher degree, only those solutions should be retained which satisfy the conditions of the problem.

EXERCISE 112

1. The sum of the squares of two numbers is 34 and their difference is one-fourth of their sum. What are the numbers?

2. The sum of the squares of two numbers is 52 and their product is 24; find the numbers.

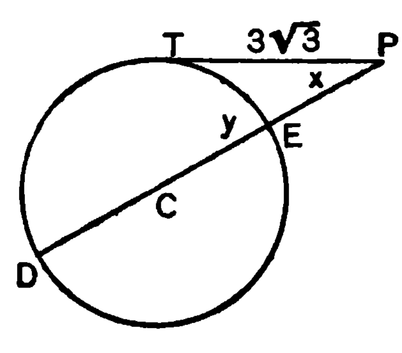
3. The sum of the sides of a triangle, ABC, is 18 inches. The sides AB and BC are equal, and the side AC is 17 less than the square of the side BC. Find the length of each side.

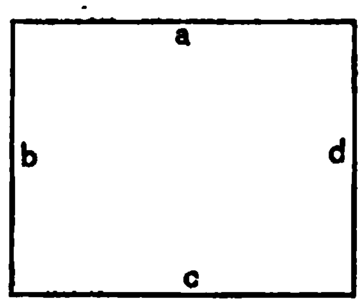


4. In a number consisting of two digits, the first digit is equal to the square of the second, and if 5 times the first digit be divided by 3 times the second, the quotient is $\frac{2}{3}$ less than twice the second digit; find the number.

5. If the length of a rectangular field were increased by 2 rods and its width diminished by 3 rods, its area would be 70 square rods; and if its length were decreased by 2 rods and its width increased by 3 rods, its area would be 110 square rods. Find the length and width.

6. A tangent TP is a mean proportional between the whole secant DP and the external segment EP. If EP equals the radius of the circle and TP is $3\sqrt{3}$, find the area of the circle.



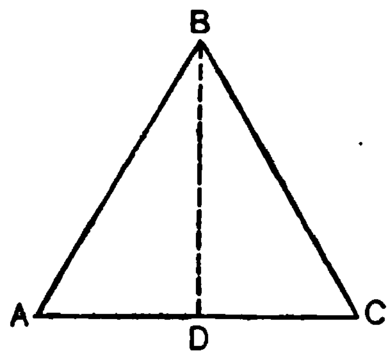


7. The perimeter, $a+b+c+d$, of a rectangle is 36, and the area of the rectangle is 80. Find the sides.

8. A farmer bought 15 cows and 20 sheep for \$720. He bought 3 more cows for \$320 than he did sheep for \$30. Find the price of each.

9. The sum of the numerator and denominator of a fraction is 7. If the numerator be diminished by 1, and the denominator be increased by 1, the product of the resulting fraction and the original fraction is $\frac{10}{3}$. Find the fraction.

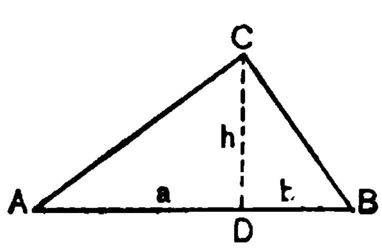
10. If 7 be added to the numerator of a fraction the value of the fraction becomes 7. If the square of the denominator be subtracted from the square of the numerator the result is 7. Find the fraction.



11. The area of a triangle ABC is one-half the product of the base, AC, and the altitude, DB. The area is 48 square feet. BC is 10 feet and its square is equal to the sum of the squares of BD and DC. $AD=DC$. Find AC and BD. Can more than one such triangle be drawn?

12. A triangle ABC has the angles B and C equal. The angle A is 60° more than the square of the number of degrees in the angle B. The sum of the three angles is 180°. Find the angles.

13. A travels from C to D. Two hours after he leaves C, B starts out to overtake him, traveling 3 miles per hour faster than A. Had A traveled 1 mile per hour slower, B would have overtaken him 12 miles nearer to C. Find A's rate.



14. In a triangle with a right angle at C, the altitude drawn from C to the hypotenuse is a mean proportional between the segments, a and b, of the hypotenuse. We

know also that $\overline{BC}^2=h^2+b^2$. If $AC=12$, $CB=9$, and $AB=15$, find a, b and h.

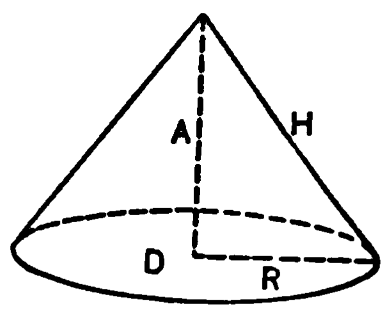
15. The sum of two numbers is to their difference as 7 is to 2. The ratio of their product is to the product of their sum and difference as 45 is to 56; find the numbers.

(Is the statement or the solution the more difficult?)

16. In a right cone, we know from geometry that

$$S=\pi RH,$$

and
$$V=\tfrac{1}{3}\pi R^2A,$$



where S = lateral surface, R = radius of base, V = volume, H = slant height, A = altitude. If $S=60\pi$ and $H=10$, find V. (Remember that because of the right angle at D, $H^2=A^2+R^2$.)

XVI. THE BINOMIAL THEOREM

POSITIVE INTEGRAL EXPONENT

249. A **Series** is a succession of terms.

A **Finite Series** is one having a limited number of terms.

An **Infinite Series** is one having an unlimited number of terms.

250. In §§ 91 and 183 we gave rules for finding the square or cube of any binomial.

The **Binomial Theorem** is a formula by means of which any power of a binomial may be expanded into a series.

251. Proof of the Binomial Theorem for a Positive Integral Exponent.

The following are obtained by actual multiplication:

$$(a+x)^2=a^2+2\,ax\;+x^2;$$
$$(a+x)^3=a^3+3\,a^2x+3\,ax^2+x^3;$$
$$(a+x)^4=a^4+4\,a^3x+6\,a^2x^2+4\,ax^3+x^4;\text{ etc.}$$

In these results, we observe the following laws:

1. The number of terms is greater by 1 than the exponent of the binomial.

2. The exponent of a in the first term is the same as the exponent of the binomial, and decreases by 1 in each succeeding term.

3. The exponent of x in the second term is 1, and increases by 1 in each succeeding term.

4. The coefficient of the first term is 1, and the coefficient of the second term is the exponent of the binomial.

5. If the coefficient of any term be multiplied by the exponent of a in that term, and the result divided by the exponent of x in the term increased by 1, the quotient will be the coefficient of the next following term.

252. If the laws of § 251 be assumed to hold for the expansion of $(a+x)^n$, where n is any positive integer, the exponent of a in the first term is n, in the second term $n-1$, in the third term $n-2$, in the fourth term $n-3$, etc.

The exponent of x in the second term is 1, in the third term 2, in the fourth term 3, etc.

The coefficient of the first term is 1; of the second term n.

Multiplying the coefficient of the second term, n, by $n-1$, the exponent of a in that term, and dividing the result by the exponent of x in the term increased by 1, or 2, we have $\frac{n(n-1)}{1\cdot 2}$ as the coefficient of the third term; and so on.

Then,
$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1\cdot 2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}x^3 + \cdots. \qquad (1)$$

Multiplying both members of (1) by $a+x$, we have

$$\begin{aligned}(a+x)^{n+1} &= a^{n+1} + na^n x + \frac{n(n-1)}{1\cdot 2}a^{n-1}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-2}x^3 + \cdots \\ &\quad + a^n x + na^{n-1}x^2 + \frac{n(n-1)}{1\cdot 2}a^{n-2}x^3 + \cdots.\end{aligned}$$

Collecting the terms which contain like powers of a and x, we have

$$(a+x)^{n+1}=a^{n+1}+(n+1)a^nx+\left[\frac{n(n-1)}{1\cdot 2}+n\right]a^{n-1}x^2$$
$$+\left[\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}+\frac{n(n-1)}{1\cdot 2}\right]a^{n-2}x^3+\cdots$$
$$=a^{n+1}+(n+1)a^nx+n\left[\frac{n-1}{2}+1\right]a^{n-1}x^2$$
$$+\frac{n(n-1)}{1\cdot 2}\left[\frac{n-2}{3}+1\right]a^{n-2}x^3+\cdots.$$

Then, $(a+x)^{n+1}=a^{n+1}+(n+1)a^nx+n\left[\frac{n+1}{2}\right]a^{n-1}x^2$
$$+\frac{n(n-1)}{1\cdot 2}\left[\frac{n+1}{3}\right]a^{n-2}x^3+\cdots$$
$$=a^{n+1}+(n+1)a^nx+\frac{(n+1)n}{1\cdot 2}a^{n-1}x^2$$
$$+\frac{(n+1)n(n-1)}{1\cdot 2\cdot 3}a^{n-2}x^3+\cdots. \qquad (2)$$

It will be observed that this result in equation (2) is of the same *form* in $n+1$, that equation (1) is in n, and equation (2) was obtained by multiplying equation (1) by $a+x$; which proves that, if the laws of § 251 hold for any power of $a+x$ whose exponent is a positive integer, they also hold for a power whose exponent is greater by 1.

But the laws have been shown to hold for $(a+x)^4$, and hence they also hold for $(a+x)^5$; and since they hold for $(a+x)^5$, they also hold for $(a+x)^6$; and so on.

Therefore, the laws hold when the exponent is any positive integer, and equation (1) is proved for every positive integral value of n.

Equation (1) is called the *Binomial Theorem.*

In place of the denominators $1\cdot 2$, $1\cdot 2\cdot 3$, etc., it is usual to write $\underline{|2}$, $\underline{|3}$, etc.

The symbol $\underline{|n}$, read "factorial-n," signifies the product of the natural numbers from 1 to n, inclusive.

The method of proof in § 252 is known as *Mathematical Induction.*

253. Putting $a=1$ in equation (1), § 252, we have

$$(1+x)^n=1+nx+\frac{n(n-1)}{\underline{|2}}x^2+\frac{n(n-1)(n-2)}{\underline{|3}}x^3+\cdots.$$

254. In expanding expressions by the Binomial Theorem, it is convenient to obtain the exponents and coefficients of the terms by aid of the laws of § 251.

1. Expand $(a+x)^5$.

The exponent of a in the first term is 5, and decreases by 1 in each succeeding term.

The exponent of x in the second term is 1, and increases by 1 in each succeeding term.

The coefficient of the first term is 1; of the second, 5.

Multiplying 5, the coefficient of the second term, by 4, the exponent of a in that term, and dividing the result by the exponent of x increased by 1, or 2, we have 10 as the coefficient of the third term; and so on.

Then, $\quad (a+x)^5=a^5+5\,a^4x+10\,a^3x^2+10\,a^2x^3+5\,ax^4+x^5.$

It will be observed that the coefficients of terms equally distant from the ends of the expansion are equal.

Thus the coefficients of the latter half of an expansion may be written out from the first half.

If the second term of the binomial is *negative*, it should be written, negative sign and all, in parentheses before applying the laws; in reducing, care must be taken to apply the principles of § 88.

2. Expand $(1-x)^6$.

$$\begin{aligned}(1-x)^6&=[1+(-x)]^6\\&=1^6+6\cdot1^5\cdot(-x)+15\cdot1^4\cdot(-x)^2+20\cdot1^3\cdot(-x)^3\\&\quad+15\cdot1^2\cdot(-x)^4+6\cdot1\cdot(-x)^5+(-x)^6\\&=1-6\,x+15\,x^2-20\,x^3+15\,x^4-6\,x^5+x^6.\end{aligned}$$

If the first term of the binomial is an arithmetical number, it is convenient to write the exponents at first without reduction; the result should afterwards be reduced to its simplest form.

If either term of the binomial has a coefficient or exponent other than unity, it should be written in parentheses before applying the laws.

3. Expand $(3\,m^2 - \sqrt[3]{n})^4$.

$$\begin{aligned}(3\,m^2 - \sqrt[3]{n})^4 &= [(3\,m^2) + (-n^{\frac{1}{3}})]^4 \\ &= (3\,m^2)^4 + 4(3\,m^2)^3(-n^{\frac{1}{3}}) + 6(3\,m^2)^2(-n^{\frac{1}{3}})^2 \\ &\qquad + 4(3\,m^2)(-n^{\frac{1}{3}})^3 + (-n^{\frac{1}{3}})^4 \\ &= 81\,m^8 - 108\,m^6 n^{\frac{1}{3}} + 54\,m^4 n^{\frac{2}{3}} - 12\,m^2 n + n^{\frac{4}{3}}.\end{aligned}$$

EXERCISE 113

1. $(c+d)^4$.
2. $(x+1)^6$.
3. $(a-b)^5$.
4. $(m-k)^8$.
5. $(ab+c^2)^5$.
6. $(x+3\,y)^4$.
7. $(2\,a-b)^5$.
8. $(4\,h+3\,k)^5$.
9. $(2\,a^4-5\,b^2)^4$.
10. $(a^{-3}-2\,b^{\frac{1}{2}})^4$.
11. $(x^{\frac{3}{5}}+2\,b^{\frac{1}{4}})^5$.
12. $(1-x^2)^8$.
13. $(2\,a^{\frac{1}{2}}+3\,b^{\frac{1}{3}})^6$.
14. $(2\,a^{\frac{1}{2}}+3\,a^{-\frac{1}{2}})^6$.
15. $\left(3\,x^{\frac{4}{5}}-\dfrac{1}{2\,x^{\frac{3}{4}}}\right)^4$
16. $(3\,a^{-\frac{1}{4}}+\sqrt[6]{a})^6$.

255. *To find the rth or general term in the expansion of* $(a+x)^n$.

The following laws hold for any term in the expansion of $(a+x)^n$, in equation (1), § 252:

1. The exponent of x is less by 1 than the number of the term.

2. The exponent of a is n minus the exponent of x.

3. The last factor of the numerator is greater by 1 than the exponent of a.

4. The last factor of the denominator is the same as the exponent of x.

Therefore in the rth term, the exponent of x will be $r-1$.

The exponent of a will be $n-(r-1)$, or $n-r+1$.

The last factor of the numerator will be $n-r+2$.

The last factor of the denominator will be $r-1$.

Hence, the rth term

$$= \frac{n(n-1)(n-2)\cdots(n-r+2)}{1\cdot 2\cdot 3\cdots(r-1)}a^{n-r+1}x^{r-1}. \qquad (1)$$

In finding any term of an expansion, it is convenient to obtain the coefficient and exponents of the terms by the above laws.

Ex. Find the 8th term of $(3a^{\frac{1}{2}}-b^{-1})^{11}$.

We have, $(3a^{\frac{1}{2}}-b^{-1})^{11}=[(3a^{\frac{1}{2}})+(-b^{-1})]^{11}$.
In this case, $n=11$, $r=8$.
The exponent of $(-b^{-1})$ is $8-1$, or 7.
The exponent of $(3a^{\frac{1}{2}})$ is $11-7$, or 4.
The first factor of the numerator is 11, and the last factor $4+1$, or 5.
The last factor of the denominator is 7.

$$\text{Then, the 8th term} = \frac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}(3a^{\frac{1}{2}})^4(-b^{-1})^7$$
$$= 330(81a^2)(-b^{-7}) = -26730a^2b^{-7}.$$

If the second term of the binomial is negative, it should be written, sign and all, in parentheses before applying the laws.

If either term of the binomial has a coefficient or exponent other than unity, it should be written in parentheses before applying the laws.

EXERCISE 114

Find the:

1. 5th term of $(c+d)^8$.
2. 5th term of $(x+1)^9$.
3. 7th term of $(a+2b)^9$.
4. 8th term of $(a^2+b^3)^{11}$.
5. 6th term of $(a^{-\frac{1}{2}}+b^{-2})^{10}$.
6. 10th term of $\left(\sqrt[5]{m^2}-\dfrac{k^{\frac{2}{3}}}{2}\right)^{14}$.
7. 5th term of $(a^4-3z^{\frac{4}{5}})^{12}$.
8. 4th term of $(c^{-3}-5cd)^{15}$.
9. Middle term of $\left(3a^5+\dfrac{b^{\frac{3}{4}}}{2}\right)^{12}$.

THE METRIC SYSTEM

Linear Measure

The standard unit of Linear Measure in the Metric System is the **Meter**. It is determined by taking one ten-millionth part of the distance from the earth's equator to either of its poles, measured on a meridian. It is equal to 39.37 inches.

The problems in this book make use of the following subdivisions of the Meter:

10 Millimeters (mm.)	= 1 Centimeter (cm.)
10 Centimeters	= 1 Decimeter (dm.)
10 Decimeters	= 1 Meter (m.)

MEASURES OF WEIGHT

The Gram is the unit of weight. It is equal to the weight of a cubic centimeter of distilled water at its greatest density.

The following multiples of the gram are used in problems in this book:

10 Grams (g.)	= 1 Dekagram (Dg.)
10 Dekagrams	= 1 Hektogram (Hg.)
10 Hektograms	= 1 Kilogram (Kg.)

XVII. HINTS ON CHECKING

256. It is sometimes desirable to check a result by numerical substitutions. Any number may be substituted for the letters involved in the problem, but since all powers of 1 are 1, a substitution of 1 for a letter above the first power is not an accurate check. It is best not to use a numerical check when other means are convenient.

In *addition:* Check: Let $a=2$, $b=2$, $c=1$.

$$\begin{array}{rcrcr}
a+2b-3c & & 2+\ 4-3 & = & 3 \\
-2a-\ b+5c & & -4-\ 2+5 & = & -1 \\
-3a-6b+7c & & -6-12+7 & = & -11 \\
\underline{9a-4b-\ c} & & \underline{18-\ 8-1} & = & \underline{\ \ 9} \\
5a-9b+8c & = & 10-18+8 & = & 0
\end{array}$$

The horizontal and vertical additions being identical is a fair, not an absolute check.

In *subtraction:* Check: Let $a=b=c=1$.

$$\begin{array}{rcrcr}
a+\ 2b-\ c & = & 1+\ 2-1 & = & 2 \\
\underline{-4a+13b+4c} & = & \underline{-4+13+4} & = & \underline{13} \\
5a-11b-5c & = & 5-11-5 & = & -11
\end{array}$$

Or add the *difference* to the *subtrahend.* The sum should be the *minuend.*

In *multiplication:* Check: Let $a=b=2$.

$$\begin{array}{ll}
2a - b & 4-2=.2 \\
\underline{3a+4b} & 6+8=14 \\
6a^2-3ab & \\
\underline{\quad +8ab-4b^2} & \\
6a^2+5ab-4b^2 & =24+\overline{20-16}=28
\end{array}$$

If the multiplicand and multiplier are homogeneous, the product will also be homogeneous, and its degree equal to the sum of the degrees of the multiplicand and multiplier.

The Illustrative examples in § 53 are instances of the above law; thus, in Ex. 2, the multiplicand, multiplier, and product are homogeneous, and of the third, first, and fourth degrees, respectively.

The student should, when possible, apply the principles of homogeneity to test the accuracy of algebraic work.

Thus, if two homogeneous expressions be multiplied together, and the product obtained is not homogeneous, it is evident that the work is not correct.

Multiplication may be checked by using the multiplier as the multiplicand and the multiplicand as the multiplier.

In *division:*

The product of the divisor and quotient should equal the dividend. If the dividend and divisor are *homogeneous*, the quotient will be homogeneous, and its degree equal to the degree of the dividend minus the degree of the divisor.

In *factoring:*

The product of the factors should equal the given expression.

In *fractions:*

Since fractions involve the four fundamental operations, addition, subtraction, multiplication, and division, the four checks above given will suffice.

In *equations:*

Reject any root which does not satisfy the *given equations.*

A SECOND COURSE IN ALGEBRA

BY

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PART I

BOSTON, U.S.A.

D. C. HEATH & CO., PUBLISHERS

1909

PREFACE

In the preparation of this text the author acknowledges joint authorship with Robert L. Short, Technical High School, Cleveland.

A knowledge of the more elementary parts of algebra is presupposed. For this reason some definitions and rules for operation are assumed as already known to the pupil.

Attention is called to the generalization and bringing together of related topics. Chapter III is an example of this feature. Here all forms of the exponent are treated. This gives opportunity to regard the logarithm as a decimal exponent and to make the logarithmic operation laws intelligible. The introduction of all linear equations and inequalities in Chapter II shows their solution directly dependent upon the four fundamental operations. It is thought that the introduction of the idea of functionality and of algebraic forms taken directly from the calculus will be found helpful to those who expect to pursue the study of mathematics further.

The treatment of factoring is thorough and so taken up that Synthetic Division becomes the natural method for factoring many higher forms and for solving equations of higher degree.

It is hoped that the treatment of variation as a proportion will remove the reluctance with which most pupils approach that subject in connection with their work in science.

In scope this text is sufficient preparation for most courses in mathematics which require thorough knowledge of the operations of algebra.

WEBSTER WELLS.

CONTENTS

ALGEBRA

I. THE FUNDAMENTAL LAWS FOR ADDITION AND MULTIPLICATION

1. The Commutative Law for Addition.

If a man gains \$8, then loses \$3, then gains \$6, and finally loses \$2, the effect on his property will be the same in whatever *order* the transactions occur.

Then, the result of adding +\$8, − \$3, + \$6, and − \$2, will be the same in whatever order the transactions occur.

Then, omitting reference to the unit, the result of adding $+8$, -3, $+6$, and -2 will be the same in whatever order the numbers are taken.

This is the Commutative Law for Addition, which is:

The sum of any set of numbers will be the same in whatever order they may be added.

2. The Associative Law for Addition.

The result of adding $b+c$ to a is expressed $a+(b+c)$, which equals $(b+c)+a$ by the Commutative Law for Addition (§ 1).

But $(b+c)+a$ equals $b+c+a$; and $b+c+a$ equals $a+b+c$, by the Commutative Law for Addition.

Whence, $$a+(b+c)=a+b+c.$$

Then, to add the sum of a set of numbers, we add the numbers separately.

This is the Associative Law for Addition.

3. The Commutative Law for Multiplication.

The product of a set of numbers will be the same in whatever order they may be multiplied.

The *sign* of the product of any number of terms is independent of their order; hence, it is sufficient to prove the commutative law for *arithmetical numbers.*

Let there be, in the figure, a stars in each row, and b rows.

We may find the entire number of stars by multiplying the number in each row, a, by the number of rows, b.

Thus, the entire number of stars is $a \times b$.

a in a row.
$\ast\ast\ast \ldots$
$\ast\ast\ast \ldots$
$\ast\ast\ast \ldots$
$\ldots\ldots\ldots$
b rows.

We may also find the entire number of stars by multiplying the number in each vertical column, b, by the number of columns, a.

Thus, the entire number of stars is $b \times a$.

Therefore, $$a \times b = b \times a,$$

which is the law for the product of two positive integers.

Again, let c, d, e, and f be any positive integers.

Then, $\frac{c}{d} \times \frac{e}{f} = \frac{c \times e}{d \times f}$; for, to multiply two fractions, we multiply the numerators together for the numerator of the product, and the denominators together for its denominator.

Then, $\frac{c}{d} \times \frac{e}{f} = \frac{e \times c}{f \times d}$; since the commutative law for multiplication holds for the product of two positive integers.

Hence, $\frac{c}{d} \times \frac{e}{f} = \frac{e}{f} \times \frac{c}{d}$; which proves the commutative law for the product of two positive fractions.

4. The Associative Law for Multiplication.

To multiply by the product of a set of numbers, we multiply by the numbers of the set separately.

The result of multiplying a by bc is expressed $a \times (bc)$, which equals $(bc) \times a$, by the Commutative Law for Multiplication.

$(bc) \times a$ equals bca, which equals abc by the Commutative Law for Multiplication.

Whence, $$a \times (bc) = abc.$$

This proves the law for the product of three numbers.

The Commutative and Associative Laws for Multiplication may be proved for the product of any number of arithmetical numbers.

(See the author's Advanced Course in Algebra, §§ 18 and 19.)

5. The Distributive Law for Multiplication.

The law is expressed $(a+b)c = ac + bc$.

We will now prove this result for all values of a, b, and c.

I. Let a and b have any values, and let c be a positive integer.

Then, $(a+b)c = (a+b) + (a+b) + \cdots$ to c terms

$= (a + a + \cdots$ to c terms$) + (b + b + \cdots$ to c terms$)$

(by the Commutative and Associative Laws for Addition),

$= ac + bc$.

II. Let a and b have any values, and let $c = \frac{e}{f}$, where e and f are positive integers.

Since the product of the quotient and divisor equals the dividend,

$$\frac{e}{f} \times f = e.$$

Then, $(a+b) \times \frac{e}{f} \times f = (a+b) \times e = ae + be$, by I.

Whence, $(a+b) \times \frac{e}{f} \times f = a \times \frac{e}{f} \times f + b \times \frac{e}{f} \times f.$

Dividing each term by f, we have

$$(a+b) \times \frac{e}{f} = a \times \frac{e}{f} + b \times \frac{e}{f}.$$

Thus, the result is proved when c is a positive integer or a positive fraction.

III. Let a and b have any values, and let $c = -g$, where g is a positive integer or fraction.

$$(a+b)(-g) = -(a+b)g = -(ag+bg), \text{ by I and II,}$$
$$= -ag - bg = a(-g) + b(-g).$$

Thus, the distributive law is proved for all positive or negative, integral or fractional, values of a, b, and c.

II. ADDITION, SUBTRACTION, MULTIPLICATION, DIVISION, APPLICATIONS

6. **Similar terms** are those which do not differ at all or differ only in their coefficients.

7. Any factor of a product may be considered the **coefficient** of the product of the remaining factors.

8. To add two similar terms, write their coefficients with the proper sign and affix the common literal part.

Ex. 1. Find the sum of ax and bx.

$$ax + bx = (a + b)x.$$

Ex. 2. Find the sum of $3\,abcx$ and $-5\,mcx$.

$$3\,abcx + (-5\,mcx) = (3\,ab - 5\,m)cx.$$

This is equivalent to taking the common factor cx from the expression

$$3\,abcx - 5\,mcx.$$

9. To subtract two similar terms find what number added to the subtrahend will produce the minuend. The number added is called the **difference**. This is equivalent to changing the sign of the subtrahend and adding the result to the minuend.

Ex. 3. Subtract $3\,ax$ from $5\,ax$. $2\,ax$ added to $3\,ax$ is $5\,ax$. Hence $2\,ax$ is the difference.

Ex. 4. From $15\,m$ take $-8\,m$. Changing the sign (mentally) of $-8\,m$, we have $15\,m + 8\,m = 23\,m$.

The written work should appear in this form:

$$\begin{array}{r} 15\,m \\ \underline{-8\,m} \\ 23\,m \end{array}$$

10. Three laws enter into multiplication of monomials:

The law of signs.

The law of coefficients.

The law of exponents.

The product of two terms of like sign is positive; the product of two terms of unlike sign is negative.

To the product of the numerical coefficients annex the letters; giving to each an exponent equal to the sum of its exponents in the factors.

The same three laws enter into division, except that *quotient* takes the place of *product* and the exponent of the divisor is subtracted from the exponent of the same letter in the dividend. (Make a rule for division of monomials.) The reason for such rule follows readily when division is defined as the process of finding one of two numbers when their product and one of the numbers are given.

11. An **equation** is a statement that two numbers are equal.

12. If an equation is true for all finite values of the unknown numbers involved, it is an **identical equation** or **identity.**

13. If an equation is true only for a definite set of values of the unknown numbers involved, it is an **equation of condition.**

14. An equation may not be true for any values of the unknowns involved. It is then said to have no roots.

15. If when a number is substituted for an unknown in an equation, the equation becomes identical (§ 12) for that number, the equation is said to be satisfied.

The **roots** of an equation are the numbers which satisfy it. A *root* of an equation is also called a **solution of the equation.**

16. Some principles used in the solution of equations are a set of generally accepted truths called **axioms.** The axioms most frequently in use are:

1. If the same number, or equal numbers, be added to equal numbers, the resulting numbers will be equal.

2. If the same number, or equal numbers, be subtracted from equal numbers, the resulting numbers will be equal.

3. If equal numbers be multiplied by the same number, or equal numbers, the resulting numbers will be equal.

4. If equal numbers be divided by the same number, or equal numbers except 0, the resulting numbers will be equal.

17. To solve an equation is to find its roots.

The following steps indicate the process:

$$\tfrac{2}{3}x - 5 = 15. \qquad (1)$$

Add 5 to each member, (Ax. 1)

$$\tfrac{2}{3}x = 15 + 5 = 20. \qquad (2)$$

Multiply each member by 3, (Ax. 3)

$$2x = 60. \qquad (3)$$

Divide each member by 2, (Ax. 4)

$$x = 30. \qquad (4)$$

18. Two equations are **equivalent** when every solution of the one is a solution of the other.

Thus equations (1), (2), (3), (4) are equivalent.

The axioms of algebra enable us to transform an equation into an equivalent one which may be more easily solved than the given one.

EXERCISE 1

1. Add $3a - 2b + 5c$, $b - 9a - 11c$, $3c + b - 2a$, $b - c - a$.

2. From the sum of $7x - 8y + 4z$ and $-2x + 5z + y$ take the sum of $x - y - z$ and $y + z - 9x$.

3. Add $3(m+n) - 5s + t$; $-8(m+n) + 4t - 11s$; $8s - 9(m+n) - 5t$; $6(m+n) - 4s + 3t$.

4. From $\frac{2}{3}p - \frac{5}{2}q + r$ take the sum of $\frac{1}{2}p + \frac{1}{8}q + \frac{2}{9}r$ and $\frac{1}{7}p - \frac{3}{4}q - \frac{1}{2}r$.

5. Subtract $ax + by + cz$ from $m^2x - y + dz$.

6. Subtract $(c-d)x - (c+d)y$ from $(2c+5d)x + (4c-3d)y$.

7. Take $mv + x$ from $md - x^2$.

8. From $4ab^2c + 5ab(c+d) - 9a^2bc^3$ take $(3a+5)b^2c - ab(c+d)$.

9. Simplify $(x^3 - 4x^2 + 5x - 1) - (2x^3 + 5x^2 - x - 7) + (x^3 + 2x^2 - 3x + 2)$.

10. Simplify $(x+1)(x-2)(x-3) - (x-2)^2 + (x^3 - 1)$.

11. Simplify $(x+y)^4 - (x-y)^4$.

12. Simplify $[4x^2 - (2x+5)][2x^2 - (x-3)]$.

13. Multiply $4x^2 + xy - y^2$ by $3x^2 - 5xy + 4y^2$.

14. Multiply $ax + by + cz$ by $bx - ay + cz$.

15. Multiply $4(m+n)^2 - 5(m+n) + 7$ by $(m+n)^2 + 2(m+n) + 1$.

16. Multiply $x^{2a+1} + x^a y^b + y^{2b}$ by $x^a - y^b$.

17. Expand $(4a + 3b)^2(4a - 3b)^2$.

18. Multiply $\frac{1}{3}a^2 - \frac{1}{4}ab + \frac{8}{5}b^2$ by $-\frac{2}{7}a + \frac{1}{3}b$.

19. Multiply $a^{2g} + a^g b^e + b^{2e}$ by $a^{2g} - a^g b^e + b^{2e}$.

20. Multiply $x^2 - xy + y^2 - xz - yz + z^2$ by $x + y + z$.

21. Multiply $x^2 + ax + bx + ab$ by $x + c$.

22. Divide $6x^6 - 19x^5 + 12x^4 + 5x^3 + 4x^2 - 6x - 2$ by $2x^2 - 3x - 1$.

23. Divide $a^{12} + b^{12}$ by $a^4 + b^4$.

24. Divide $32m^5 - 243n^5$ by $2m - 3n$.

25. Divide $\frac{1}{125}a^3 + \frac{8}{27}b^3$ by $\frac{1}{5}a + \frac{2}{3}b$.

26. Divide $a^{6n} - b^{6n}$ by $a^{2n} + a^n b^n + b^{2n}$.

27. Divide $\frac{1}{6}x^3 + \frac{7}{36}x^2y + \frac{1}{8}y^3$ by $\frac{1}{3}x + \frac{1}{2}y$.

28. Divide $9r^2s^2 + 15r^4 - 38r^3s - 8s^4 - 26rs^3$ by $5r^2 + 4s^2 - rs$.

29. Divide $7m^{2x+4} - 8m^{x+2}n^{2x-1} - 12n^{4x-2}$ by $m^{x+2} - 2n^{2x-1}$.

30. Divide $x^3 + (a+b)x^2 - (6a^2 - 5ab)x + 6a^2b$ by $x + 3a$.

Solve the following equations and verify results:

31. $(x+2)(x-5) = x^2 - 4x - 4$.

32. $6(x-3)+5(4x-7)+1=0.$

33. $\frac{3}{2}v-4+\frac{5}{8}v-\frac{1}{5}v=\frac{7}{8}v-\frac{1}{5}.$

34. $\frac{2}{5}(3x-2)-\frac{1}{2}(3x-2)=\frac{1}{7}(3x-2)-17.$

35. $(y-4)(y+3)(y-2)=(y-1)^3-1.$

36. $\dfrac{6t+5}{15}-\dfrac{13}{21}=\dfrac{2t}{5}+\dfrac{t}{3}.$

37. $ab+ax+3b^2-2a^2=4bc-bx+cx-c^2-ac.$ Solve for x.

38. $y-e=-\dfrac{1}{m}(x-d).$ Solve for x.

39. $(a+b+c)(x-2a)-(x-c)(a+b)$
$=(a-b-c)^2-(a^2+b^2).$

40. $\dfrac{ax-b}{a}+\dfrac{bx-c}{b}+\dfrac{cx-a}{c}=0.$

19. It is sometimes convenient to indicate operations of addition and subtraction. For this purpose parentheses are used. The various forms of parentheses are: *parentheses* ( ), *braces* { }, *brackets* [ ], and the *vinculum* $\overline{\quad\quad}$.

A *positive sign* before parentheses indicates that the number within is to be added. Hence, parentheses preceded by a + sign may be removed without changing the signs of the terms within.

Ex. $2a+3b+(3a-5b+c)=2a+3b+3a-5b+c.$

A *negative sign* before parentheses indicates that the number within the parentheses is to be subtracted. Hence, parentheses preceded by a − sign may be removed if the + signs of the terms within be changed to − and the − signs to + (§ 9).

Ex. 1. $5a+3b-(4a+7b)=5a+3b-4a-7b=a-4b.$

Ex. 2. $5a+3b-(-4a+7b)=5a+3b+4a-7b=9a-4b.$

If the expression contains two or more parentheses, one within the other, remove one at a time beginning with the inner parentheses.

Ex.

$$\begin{aligned}&5a+\{3a-(5b+2a)\}=\\&5a+\{3a-5b-2a\}=\\&5a+3a-5b-2a=\\&\quad 6a-5b.\end{aligned}$$

EXERCISE 2

Simplify the following by removing the signs of aggregation, and then uniting similar terms:

1. $9m+(-4m+6n)-(3m-n)$.
2. $2x-3y-[5x+y]+\{-8x-7y\}$.
3. $4y^2-2x^2-[-4x^2-7xy+5y^2]+(8x^2-9xy)$.
4. $3a^2-5ab-\{-4a^2+2ab-9b^2\}-\overline{7a^2-6ab+b^2}$.
5. $5a-(7a-[9a+4])$.
6. $7x-\{-8y-\overline{10x-11y}\}$.
7. $6mn+5-([-7mn-3]-\{-5mn-11\})$.
8. $2a-(-3b+c-\{a-b\})-(3a+2c-[-2b+3c])$.
9. $37-[41-\{13-(56-\overline{28+7})\}]$.
10. $9m-(3n+\{4m-[n-6m]\}-[m+7n])$.

11. In each of the above expressions find the value if $a=1$, $b=-2$, $c=-3$, $m=5$, $n=2$, $x=-4$, $y=-1$.

20. A number may be enclosed in parentheses preceded by a + sign without changing the sign of its terms, but if a number is enclosed in parentheses preceded by a − sign, each plus term placed in parentheses is changed to *minus* and each *minus* term to *plus*.

EXERCISE 3

In each of the following expressions, enclose the last three terms in parentheses preceded by a − sign:

1. $a-b-c+d$.
2. m^3+2m^2+3m+4.
3. $x+x^2y-xy^2-y^3$.
4. $a^2-4b^2+12b-9$.

5. $4x^2 - y^2 - 2yz - z^2$.

6. $a^2 + b^2 - c^2 + d^2$.

7. $x^2 - 2xy + y^2 + 3x - 4y$.

8. $n^4 - 5n^3 - 8n^2 + 6n + 7$.

DEGREE OF A RATIONAL EXPRESSION

21. A monomial is said to be **rational and integral** when it is either a number expressed in Arabic numerals, or a single letter with unity for its exponent, or the product of two or more such numbers or letters.

Thus, $3a^2b^3$, being equivalent to $3 \cdot a \cdot a \cdot b \cdot b \cdot b$, is rational and integral.

A polynomial is said to be rational and integral when each term is rational and integral; as $2x^2 - \frac{3}{4}ab + c^3$.

22. If a term has a literal portion which consists of a single letter with unity for its exponent, the term is said to be of the *first degree.*

Thus, $2a$ is of the first degree.

The **degree** of any rational and integral monomial (§ 21) is the number of terms of the first degree which are multiplied together to form its literal portion.

Thus, $5ab$ is of the *second* degree; $3a^2b^3$, being equivalent to $3 \cdot a \cdot a \cdot b \cdot b \cdot b$, is of the *fifth* degree; etc.

The degree of a rational and integral monomial equals the sum of the exponents of the letters involved in it.

Thus, ab^4c^3 is of the *eighth* degree.

The degree of a rational and integral polynomial is the degree of its term of highest degree.

Thus, $2a^2b - 3c + d^2$ is of the *third* degree.

23. If a rational and integral monomial (§ 21) involves a certain letter, its *degree with respect to it* is denoted by its exponent.

If it involves two letters, its *degree with respect to them* is denoted by the sum of their exponents; etc.

Thus, $2ab^4x^2y^3$ is of the *second* degree with respect to x and of the *fifth* with respect to x and y.

24. An **Integral Equation** is one each of whose members is a rational and integral expression (§ 21); as,

$$4x - 5 = \tfrac{2}{3}y + 1.$$

A **Numerical Equation** is one in which all the known numbers are represented by Arabic numerals; as,

$$2x - 7 = x + 6.$$

25. If an integral equation (§ 24) contains one or more unknown numbers, the *degree* of the equation is the degree of its term of highest degree.

Thus, if x and y represent unknown numbers,

$ax - by = c$ is an equation of the *first* degree;

$x^2 + 4x = -2$, an equation of the *second* degree;

$2x^2 - 3xy^2 = 5$, an equation of the *third* degree; etc.

A **Linear,** or **Simple, Equation** is an equation of the first degree.

26. The equations of Exercise 1 were integral, first degree in one unknown number, linear.

THEOREMS IN REGARD TO EQUIVALENT EQUATIONS

27. If the same expression be added to both members of an equation, the resulting equation will be equivalent to the first.

Let $$A = B \qquad (1)$$

be an equation involving one or more unknown numbers.

To prove the equation $$A + C = B + C, \qquad (2)$$

where C is any expression, equivalent to (1).

Any solution of (1), when substituted for the unknown numbers, makes A identically equal to B (§ 15).

It then makes $A + C$ identically equal to $B + C$ (§ 16, 1).

Then it is a solution of (2).

5. $4x^2 - y^2 - 2yz - z^2$.
6. $a^2 + b^2 - c^2 + d^2$.
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8. $n^4 - 5n^3 - 8n^2 + 6n + 7$.

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THEOREMS IN REGARD TO EQUIVALENT EQUATIONS

27. If the same expression be added to both members of an equation, the resulting equation will be equivalent to the first.

Let $$A = B \qquad (1)$$

be an equation involving one or more unknown numbers.

To prove the equation $$A + C = B + C, \qquad (2)$$

where C is any expression, equivalent to (1).

Any solution of (1), when substituted for the unknown numbers, makes A identically equal to B (§ 15).

It then makes $A + C$ identically equal to $B + C$ (§ 16, 1).

Then it is a solution of (2).

It follows from this that it is never allowable to divide both members of an integral equation by an expression which involves the unknown numbers; for in this way solutions are lost.

33. If both members of a fractional equation be multiplied by the L.C.M. of the given denominators, the resulting equation is in general equivalent to the first.

Let all the terms be transposed to the first member, and let them be added, using for a common denominator the L. C. M. of the given denominators.

The equation will then be in the form

$$\frac{A}{B} = 0. \tag{1}$$

We will now prove the equation

$$A = 0, \tag{2}$$

which is obtained by multiplying (1) by the L. C. M. of the given denominators, equivalent to (1), *if A and B have no common factor.*

Any solution of (1), when substituted for the unknown numbers, makes $\frac{A}{B}$ identically equal to 0.

Then, it must make A identically equal to 0.

Then, it is a solution of (2).

Again, any solution of (2), when substituted for the unknown numbers, makes A identically equal to 0.

Since A and B have no common factor, B cannot be 0 when this solution is substituted for the unknown numbers.

Then, any solution of (2), when substituted for the unknown numbers, makes $\frac{A}{B}$ identically equal to 0, and is a solution of (1).

Therefore, (1) and (2) are equivalent, if A and B have no common factor.

If A and B have a common factor, (1) *and* (2) *are not equivalent;* consider, for example, the equations

$$\frac{x-1}{x^2-1} = 0, \text{ and } x - 1 = 0.$$

The second equation is satisfied by the value $x = 1$, which does not satisfy the first equation; then, the equations are not equivalent.

34. A fractional equation may be cleared of fractions by multiplying both members by *any* common multiple of the denominators; but in this way additional solutions are often introduced, and the resulting equation is not equivalent to the first.

Consider, for example, the equation

$$\frac{x^2}{x^2-1}+\frac{x}{x-1}=2.$$

If we solve by multiplying both members by x^2-1, the L. C. M. of x^2-1 and $x-1$, we find $x=-2$.

If, however, we multiply both members by $(x^2-1)(x-1)$, we have

$$x^3-x^2+x^3-x=2x^3-2x^2-2x+2, \text{ or } x^2+x-2=0.$$

The latter equation may be solved by using factors.

The factors of x^2+x-2 are $x+2$ and $x-1$.

Solving the equation $x+2=0$, $x=-2$.

Solving the equation $x-1=0$, $x=1$.

This gives the additional value $x=1$; and it is evident that this does not satisfy the given equation.

35. If both members of an equation be raised to the same positive integral power (§ 66), the resulting equation will have all the solutions of the given equation, and, in general, additional ones.

Consider, for example, the equation $x=3$.

Squaring both members, we have

$$x^2=9, \text{ or } x^2-9=0, \text{ or } (x+3)(x-3)=0.$$

The latter equation has the root 3, and, in addition, the root -3.

We will now consider the general case.

Let
$$A=B \tag{1}$$
be an equation involving one or more unknown numbers.

Raising both members to the nth power, n being a positive integer, we have
$$A^n=B^n, \text{ or } A^n-B^n=0. \tag{2}$$

Factoring the first number (§ 103, VII),
$$(A-B)(A^{n-1}+A^{n-2}B+\cdots+B^{n-1})=0. \tag{3}$$

Now, equation (3) is satisfied when $A=B$.

Whence, equation (2) has all the solutions of (1).

But (3) is also satisfied when

$$A^{n-1} + A^{n-2}B + \cdots + B^{n-1} = 0;$$

so that (2) has also the solutions of this last equation, which, in general, do not satisfy (1).

EQUIVALENT SYSTEMS OF EQUATIONS

36. Two systems of equations, involving two or more unknown numbers, are said to be *equivalent* when every solution of the first system is a solution of the second, and every solution of the second is a solution of the first.

37. If
$$\begin{cases} A = 0, \\ B = 0, \end{cases}$$

are equations involving two or more unknown numbers, the system of equations

$$\begin{cases} A = 0, \\ mA + nB = 0, \end{cases}$$

where m and n are any numbers, and n not equal to zero, is equivalent to the first system.

For any solution of the first system, when substituted for the unknown numbers, makes $A = 0$ and $B = 0$.

It then makes $A = 0$ and $mA + nB = 0$.

Then, it is a solution of the second system.

Again, any solution of the second system, when substituted for the unknown numbers, makes $A = 0$ and $mA + nB = 0$.

It therefore makes $nB = 0$, or $B = 0$.

Since it makes $A = 0$ and $B = 0$, it is a solution of the first system.

Hence, the systems are equivalent.

A similar result holds for a system of any number of equations.

Either m or n may be *negative*.

38. If either equation, in a system of two, be solved for one of the unknown numbers, and the value found be substituted for this unknown number in the other equation, the resulting system will be equivalent to the first.

Let
$$\begin{cases} A = B, & (1) \\ C = D, & (2) \end{cases}$$

be equations involving two unknown numbers, x and y.

Let E be the value of x obtained by solving (1).

Let $F = G$ be the equation obtained by substituting E for x in (2).

To prove the system of equations

$$\begin{cases} x = E, & (3) \\ F = G, & (4) \end{cases}$$

equivalent to the first system.

Any solution of the first system satisfies (3), for (3) is only a form of (1).

Also, the values of x and y which form the solution make x and E equal; and hence satisfy the equation obtained by putting E for x in (2).

Then, any solution of the first system satisfies (4).

Again, any solution of the second system satisfies (1), for (1) is only a form of (3).

Also, the values of x and y which form the solution make x and E equal; and hence satisfy the equation obtained by putting x for E in (4).

Then, any solution of the second system satisfies (2).

Hence, the systems are equivalent.

A similar result holds for a system of any number of equations, involving any number of unknown numbers.

39. The principles of §§ 27, 28, 29, 31, 33, 35, 36, and 37 hold for equations of any degree.

40. In the solution of an equation of Exercise 1, we replaced each equation by an equivalent one more easily solved for the unknown number.

41. **Elimination** is the process of deriving from a system of two or more equations, a system containing one less unknown number than the given system.

There are several methods of elimination, each method depending on a process which will form a second system equivalent to the first.

42. A system of equations is called **Simultaneous** when each contains two or more unknown numbers, and every equation of the system is satisfied by the same set, or sets, of values of the unknown numbers; thus, each equation of the system

$$\begin{cases} x + y = 6, \\ x - y = 3, \end{cases}$$

is satisfied by the set of values $x = 4$, $y = 1$.

A **Solution** of a system of simultaneous equations is a set of values of the unknown numbers which satisfies every equation of the system; to *solve* a system of simultaneous equations is to find its solutions.

Ex. Solve

$$\left.\begin{array}{ll}(1) & 2x+5y=9,\\ (2) & x-y=1,\end{array}\right\} \quad \text{I.}$$

$$\left.\begin{array}{ll}(1) & 2x+5y=9,\\ (3) & 5x-5y=5,\end{array}\right\} \quad \text{II.}$$

$$\left.\begin{array}{lr}(1) & 2x+5y=9,\\ (4) & (2x+5y)+5x-5y=9+5,\end{array}\right\}, \text{ or } \left.\begin{array}{r}2x+5y=\ 9,\\ 7x=14,\end{array}\right\} \quad \text{III.}$$

System II is equivalent to system I, and system III is equivalent to system II.

System III gives the required solution since (4) gives $x=2$ and this value substituted in (1) gives $y=1$.

Similarly it may be shown that elimination by substitution and by comparison involve the deriving of equivalent systems from the given system (§§ 37, 38).

Unless the equations of a given system are independent a solution is not possible.

43. If two equations, containing two or more unknown numbers, are not equivalent, they are called **Independent.**

Consider the equations

$$\begin{cases} x+y=5, & (1)\\ x+y=6. & (2)\end{cases}$$

It is evidently impossible to find a set of values of x and y which shall satisfy both (1) and (2).

Such equations are called **Inconsistent.**

EXERCISE 4

Solve the following equations, using Addition or Subtraction, Substitution or Comparison:

1. $\begin{cases}3x+5y=21.\\ 7x-2y=\ 8.\end{cases}$

2. $\begin{cases}x-2y=9.\\ 2x-y=12.\end{cases}$

3. $\begin{cases} 4x-3y=1. \\ 6x+15y=8. \end{cases}$

4. $\begin{cases} 2x-y=-3. \\ 6x+9=3y. \end{cases}$

5. $\begin{cases} y=4+x. \\ 3x-3y=-12. \end{cases}$

6. $\begin{cases} \frac{2}{3}x+\frac{1}{2}y=7. \\ 3x-\frac{1}{6}y=17. \end{cases}$

7. $\begin{cases} 3x-2y=1\frac{1}{2}. \\ x=2y. \end{cases}$

8. $\begin{cases} \frac{1}{2}m+\frac{2}{3}n=-2. \\ 3m+12=-4n. \end{cases}$

9. $\begin{cases} s+4v=-1. \\ v=2s-16. \end{cases}$

10. $\begin{cases} \dfrac{x+2}{7}=\dfrac{y}{5}-1. \\ \dfrac{2x-9}{3}+\dfrac{2y-5}{5}=10. \end{cases}$

11. $\begin{cases} \dfrac{17p-q}{7}=p-3q. \\ 8p+q=15. \end{cases}$

12. $\begin{cases} 3x+\dfrac{x-y}{3}=25. \\ 15-2x+\dfrac{y}{5}=0. \end{cases}$

13. $\begin{cases} 11t=u+19. \\ 2t-u=10. \end{cases}$

14. $\begin{cases} \dfrac{2}{3}(x-3y)-\dfrac{2x-y}{2}=-5. \\ \dfrac{2x+3y-6}{9}-\dfrac{x-y}{7}=1. \end{cases}$

15. $\begin{cases} \dfrac{6x-5y+10}{11}-\dfrac{5x+3y}{7}=\dfrac{4}{15}. \\ 5y-3x-1=0. \end{cases}$

16. $\begin{cases} y-3x=a. \\ \frac{3}{2}x+\frac{6}{7}y=9a. \end{cases}$

17. $\begin{cases} \dfrac{x+y+2}{4}-\dfrac{3x-y}{17}-5=\dfrac{x}{6}. \\ \dfrac{x}{9}+\dfrac{y}{5}=6. \end{cases}$

18. $\begin{cases} \dfrac{x+y-1}{27}-\dfrac{1}{5}(x-y)=1. \\ 4y=17\frac{1}{2}+\frac{11}{4}x. \end{cases}$

19. $\begin{cases} 2\,ax - 4\,by = a^2 - ab + 2\,b^2. \\ x + y = a. \end{cases}$

20. $\begin{cases} \dfrac{x+y}{4} = c + d. \\ 2\,x - y = 5\,c - 7\,d. \end{cases}$

21. $\begin{cases} \dfrac{x}{17} + \dfrac{y}{15} = 10. \\ \dfrac{x}{5} - \dfrac{y}{12} = 10\frac{3}{4}. \end{cases}$

22. If 5 in. be added to the length and 3 in. to the breadth of a certain rectangle, the area is increased by 120 sq. in., but if 4 in. be subtracted from the length and 2 in. from the breadth, the area is decreased by 70 sq. in. Find its dimensions.

23. 2 cu. ft. of water and 4 cu. ft. of ice together weigh 355 lb. The difference between the weights of 3 cu. ft. of water and 2 cu. ft. of ice is 72 lb. 8 oz. Find the weights of a cubic foot of each.

24. A masonry contractor held back \$132.50 of the wages due his men. His bricklayers earned \$3 per day, and his hod carriers \$1.75 per day. Their combined wages for a day were \$256.25. He retained \$1.50 from each bricklayer and \$1 from each hod carrier. How many carriers did he employ?

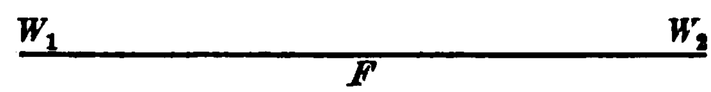
25. A man rows a certain distance down stream at the rate of $3\frac{1}{2}$ mi. an hour in $3\frac{1}{2}$ hr. In returning it takes him 16 hr. to reach a point 5 mi. below his starting point. Find the rate of the current.

26. Two trains start toward each other, one from New York, the other from Chicago. They meet in 10 hr., 40 min., the distance between the two cities being 960 mi. If the first train starts 3 hr. earlier than the second train, they will meet $9\frac{1}{3}$ hr. after the second train starts. Find the rate of each train.

27. A number lies between 300 and 400. If 18 is added to the number, the last two digits change places with each other, and if the number be divided by the number expressed by the first two digits, the quotient is $10\frac{3}{17}$. Find the number.

28. Find two numbers whose difference is 93 and whose sum divided by the smaller number gives a quotient of $6\frac{3}{7}$.

29. By the law of levers, the product of the weight W_1 by the distance from W_1 to the fulcrum, F, is equal to the product of the weight W_2 by the distance from W_2 to the fulcrum.



A board resting across a pole balances when a 60-lb. boy is on one end and a 100-lb. boy on the other end. The board will also balance if a 120-lb. boy sits 2 ft. from one end and a 60-lb. boy sits 2 ft. from the other end. Find the length of the board.

30. If a regular hexagon is circumscribed about a given circle, the difference between the areas of the hexagon and circle is 32.24, and the sum of their areas is 660.56. Find the radius of the circle.

GRAPHICAL REPRESENTATION

44. A drawing or picture of given data or of an equation is often of value.

45. Descartes (1596–1650) was the first mathematician to apply measurement to equations.

It is impossible to locate absolutely a point in a plane. All measurements are purely relative, and all positions in a plane or in space are likewise relative. Since a plane is infinite in length and breadth, it is necessary to have some fixed form from which one can take measurements. For this form, assumed fixed in a plane, Descartes chose two intersecting lines as a coördinate system. Such a system of coördinates has since his time been called Cartesian. It will best suit our purpose to choose lines intersecting at right angles.

46. The Point. If we take any point M, its position is determined by the length of the lines $QM = x$ and $PM = y$, parallel to the intersecting lines OX and OY (Fig. 1). The values $x = a$ and $y = b$ will thus determine a point. The unit of length can be arbitrarily chosen, but when once fixed remains

the same throughout the problem under discussion. $QM = x$ and $PM = y$, we call the *coördinates* of the point M. x, measured parallel to OX, is called the *abscissa*. y, measured parallel to OY, is the *ordinate*. OX and OY are the *coördinate axes*. OX is the axis of x, also called the axis of abscissas. OY is the axis of y, also called the axis of ordinates. O, the point of intersection, is called the *origin*.

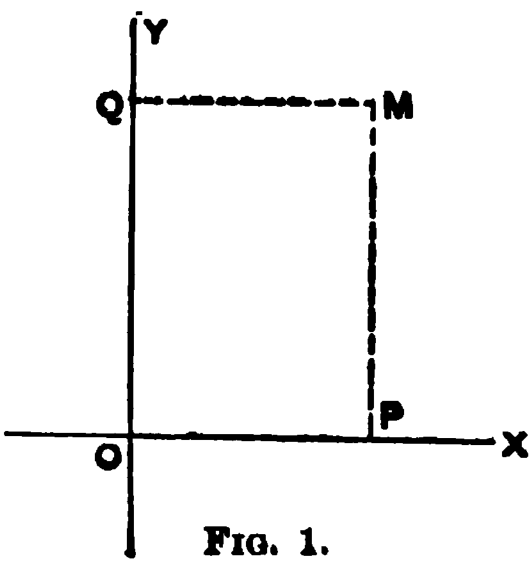


FIG. 1.

Two measurements are necessary to locate a point in a plane.

For example, $x = 2$ holds for any point on the line AB (Fig. 2). But if in addition we demand that $y = 3$, the point is fully determined by the intersection of the lines AB and CD, any point on CD satisfying the equation $y = 3$.

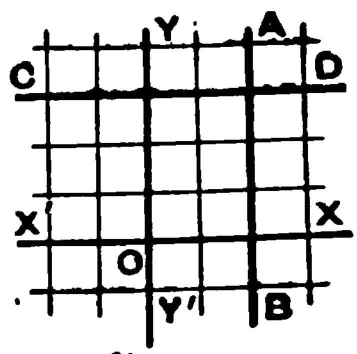


FIG. 2.

47. The Line. Consider the equation $x + y = 6$.

In this equation, when values are assigned to x, we get a value of y for every such value of x. When $x = 0$, $y = 6$; $x = 1$, $y = 5$; $x = 2$, $y = 4$; $x = 3$, $y = 3$; $x = 5$, $y = 1$; etc., giving an infinite number of values of x and y which satisfy the equation.

Laying off these values on a pair of axes, as shown in § 46, we see that the points whose coördinates satisfy this equation lie on the line AB (Fig. 3). It is readily seen that there might be confusion as to the direction from the origin in which the measurements should be taken. This is avoided by a simple convention in signs. Negative values of x are measured to the left of the y-axis, positive to the right. In like manner, negative values of y are measured downward from the x-axis, positive values upward. XOY, YOX', $X'OY'$, $Y'OX$, are spoken of as the first, second, third, and fourth quadrants respectively. (See Fig. 2.)

By plotting other equations of the *first degree* with two unknown quantities it will be seen that such an equation always

represents a *straight line*. This line AB (Fig. 3) is called the graph of $x+y=6$ and is the locus of all the points satisfying that equation.

48. Now plot two simultaneous equations of the first degree on the same axes, *e.g.* $x+y=6$ and $2x-3y=-3$ (Fig. 4). We see that the coördinates of the point of intersection have the same values as the x and y of the algebraic solution of the equations.

This is a geometric or graphical reason why there is but one solution to a pair of simultaneous equations of the first degree with two unknown numbers. A simple algebraic proof will be given in the next article. Hereafter an equation of the first degree in two variables will be called a *linear equation*.

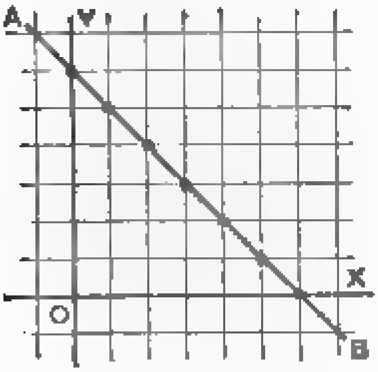


FIG. 3.

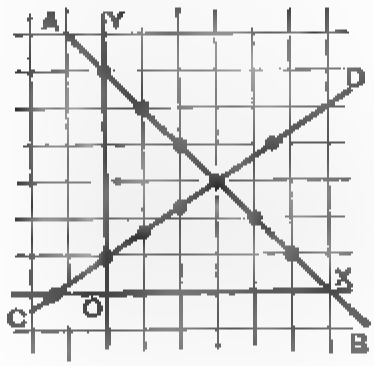


FIG. 4.

49. Algebraic Proof of the Principle of § 48. Two simultaneous equations of the first degree cannot be satisfied by two different sets of values for x and y. Given the equations

$$ax+by=c, \tag{1}$$

$$ex+fy=h. \tag{2}$$

Eliminating y, $$(af-eb)x=cf-bh. \tag{3}$$

Let x_1 and x_2 be the roots of (3), different in value. Substituting these roots, we have

$$(af-eb)x_1=cf-bh,$$
$$(af-eb)x_2=cf-bh,$$
$$(af-eb)(x_1-x_2)=0.$$

But $x_1 \neq x_2$, $\therefore af=eb$, or $\frac{a}{e}=\frac{b}{f}$, which is impossible.

In general, the plotting of two graphs on the same axes will determine all the *real* solutions of the two equations, the

coördinates of each point of intersection of the graphs being values of x and y which satisfy both equations.

50. It is well to introduce the subject of graphs by the use of concrete problems which depend on two conditions and which can be solved without mention of the word *equation*.

Professor F. E. Nipher, Washington University, St. Louis, proposes the following:

"A person wishing a number of copies of a letter made, went to a typewriter and learned that the cost would be, for mimeograph work:

(1)
- \$1.00 for 100 copies,
- \$2.00 for 200 copies,
- \$3.00 for 300 copies,
- \$4.00 for 400 copies, and so on.

"He then went to a printer and was made the following terms:

(2)
- \$2.50 for 100 copies,
- \$3.00 for 200 copies,
- \$3.50 for 300 copies,
- \$4.00 for 400 copies, and so on, a rise of 50 cents for each hundred.

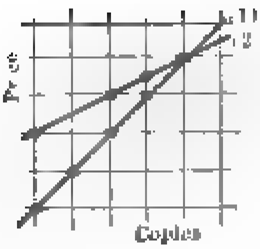


FIG. 5.

"Plotting the data of (1) and (2) on the same axes, we have:

"The vertical axis being chosen for the price-units, the horizontal axis for the number of copies.

"Any point on line (1) will determine the price for a certain number of mimeograph copies. Any point on line (2) determines the price and corresponding number of copies of printer's work."

Numerous lessons can be drawn from this problem. One is that for less than 400 copies, it is less expensive to patronize the mimeographer. For 400 copies, it does not matter which party is patronized. For *no* copies from the mimeographer, one pays nothing. How about the cost of *no* copies from the printer? Why?

The graph offers an excellent method for the solution of indeterminate equations in positive integers.

Ex. Solve $3x + 4y = 22$ for positive integers. Plotting the equation, we have

x	y
0	$5\frac{1}{2}$
2	4
3	$3\frac{1}{4}$
6	1

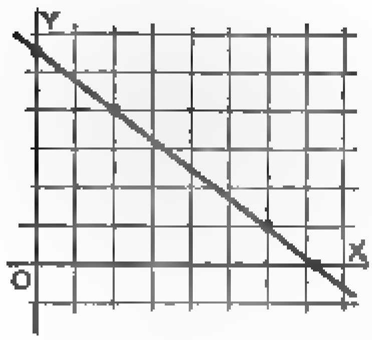


FIG. 6.

We see that the line crosses the corner of a square only when $x=2$ and $x=6$. For all other integral values of x, y is fractional. The only positive integral solutions are, therefore, $x=2$, $y=4$; $x=6$, $y=1$. This corresponds to the algebraic result.

51. In the equation $y=\frac{22-3x}{4}$, y is dependent on x for its value. That is, every change in x produces a change in y. When two quantities are so related, the first is said to be a **Function** of the second. Similarly $y=f(x)$, read y is a function of x, means that y is equal to some expression in x. In place of the equation represented by Fig. 6 one might have

$$f(x)=\frac{22-3x}{4}.$$

$$\begin{cases} f(x)=8-2x, \\ F(x)=4+x.^* \end{cases}$$

Make a graph of each of these two functions and find their point of intersection.

EXERCISE 5

1. $f(x)=7x-24$, find $f(0)$, $f(1)$, $f(2)$, $f(-4)$, $f(3\frac{3}{7})$.
2. $\phi(x)=x^2-2x+1$, find $\phi(0)$, $\phi(1)$, $\phi(2)$.

* The $f(x)$ and $F(x)$ mean simply different functions of x. In these same equations $f(0)$ means the value of the function when 0 is substituted for x in $f(x)=8-2x$, namely, $f(0)=8$. Similarly $f(1)=8-2(1)=6$.

Solve the following by means of graphs:

3. $\begin{cases} 2x - 5y = -16, \\ 3x + 7y = 5. \end{cases}$

4. $\begin{cases} \dfrac{x-5}{4} - \dfrac{2x-y-1}{3} = \dfrac{2y-2}{5}. \\ \dfrac{2y+x-1}{9} = \dfrac{x+y}{4}. \end{cases}$

5. $\begin{cases} f(y) = \dfrac{3y-62}{7}. \\ F(y) = \dfrac{2y-44}{5}. \end{cases}$

6. $\begin{cases} \phi(y) = \dfrac{1-8y}{15}. \\ \psi(y) = \dfrac{7y-24}{10}. \end{cases}$

7. $\begin{cases} f(x) = \dfrac{5x-19}{3}. \\ F(x) = \dfrac{2-7x}{4}. \end{cases}$

INEQUALITIES

52. The **Signs of Inequality**, $>$ and $<$, are read "*is greater than*" and "*is less than*," respectively.

Thus, $a > b$ is read "a is greater than b"; $a < b$ is read "a is less than b."

53. One number is said to be *greater* than another when the remainder obtained by subtracting the second from the first is a *positive* number.

One number is said to be *less* than another when the remainder obtained by subtracting the second from the first is a *negative* number.

Thus, if $a - b$ is a positive number, $a > b$; and if $a - b$ is a negative number, $a < b$.

54. An **Inequality** is a statement that one of two expressions is greater or less than another.

The *First Member* of an inequality is the expression to the left of the sign of inequality, and the *Second Member* is the expression to the right of that sign.

Any term of either member of an inequality is called a *term* of the inequality.

55. Two or more inequalities are said to *subsist in the same sense* when the first member is the greater or the less in both.

Thus, $a > b$ and $c > d$ subsist in the same sense.

PROPERTIES OF INEQUALITIES

56. An inequality will continue in the same sense after the same number has been added to, or subtracted from, both members.

For consider the inequality $a > b$.

By § 53, $a - b$ is a positive number.

Hence, each of the numbers

$$(a + c) - (b + c), \text{ and } (a - c) - (b - c)$$

is positive, since each is equal to $a - b$.

Therefore, $\quad a + c > b + c$, and $a - c > b - c$. (§ 53)

57. It follows from § 56 that **a term may be transposed from one member of an inequality to the other by changing its sign.**

If the same term appears in both members of an inequality, affected with the same sign, it may be removed.

58. If the signs of all the terms of an inequality be changed, the sign of inequality must be reversed.

For consider the inequality $\quad a - b > c - d.$

Transposing every term, $\quad d - c > b - a.$ (§ 57)

That is, $\quad b - a < d - c.$

59. An inequality will continue in the same sense after both members have been multiplied or divided by the same positive number.

For consider the inequality $a > b$.

By § 53, $a - b$ is a positive number.

Hence, if m is a positive number, each of the numbers

$$m(a - b) \text{ and } \frac{a - b}{m}, \text{ or } ma - mb \text{ and } \frac{a}{m} - \frac{b}{m}, \text{ is positive.}$$

Therefore, $\quad ma > mb$, and $\dfrac{a}{m} > \dfrac{b}{m}$.

60. It follows from §§ 58 and 59 that **if both members of an inequality be multiplied or divided by the same negative number, the sign of inequality must be reversed.**

61. If any number of inequalities, subsisting in the same sense, be added member to member, the resulting inequality will also subsist in the same sense.

For consider the inequalities $a > b$, $a' > b'$, $a'' > b''$, $\cdots$.

Each of the numbers, $a - b$, $a' - b'$, $a'' - b''$, $\cdots$, is positive.

Then, their sum $a - b + a' - b' + a'' - b'' + \cdots$,

or $$a + a' + a'' + \cdots - (b + b' + b'' + \cdots),$$

is a positive number.

Whence, $$a + a' + a'' + \cdots > b + b' + b'' + \cdots.$$

If two inequalities, subsisting in the same sense, be *subtracted* member from member, the resulting inequality does not necessarily subsist in the same sense.

Thus, if $a > b$ and $a' > b'$, the numbers $a - b$ and $a' - b'$ are positive.

But $(a - b) - (a' - b')$, or its equal, $(a - a') - (b - b')$, may be positive, negative, or zero; and hence $a - a'$ may be greater than, less than, or equal to $b - b'$.

62. If $a > b$ and $a' > b'$, and each of the numbers a, a', b, b', is positive, then
$$aa' < bb'.$$

Since $a' > b'$, and a is positive,
$$aa' > ab' \text{ (§ 59).} \qquad (1)$$

Again, since $a > b$, and b' is positive,
$$ab' > bb'. \qquad (2)$$

From (1) and (2), $$aa' > bb'.$$

63. If we have any number of inequalities subsisting in the same sense, as $a > b$, $a' > b'$, $a'' > b''$, $\cdots$, and each of the numbers a, a', a'', $\cdots$, b, b', b'', $\cdots$, is positive, then
$$aa'a'' \cdots > bb'b'' \cdots.$$

For by § 62, $$aa' > bb'.$$

Also, $$a'' > b''.$$

Then by § 62, $$aa'a'' > bb'b''.$$

Continuing the process with the remaining inequalities, we obtain finally
$$aa'a'' \cdots > bb'b'' \cdots.$$

64. Examples.

1. Find the limit of x in the inequality
$$7x - \frac{23}{3} < \frac{2x}{3} + 5.$$

Multiplying both members by 3 (§ 59), we have

$$21x - 23 < 2x + 15.$$

Transposing (§ 57), and uniting terms,

$$19x < 38.$$

Dividing both members by 19 (§ 59),

$$x < 2.$$

(This means that, for any value of $x < 2$, $7x - \frac{23}{3} < \frac{2x}{3} + 5$.)

2. Find the limits of x and y in the following:

$$\begin{cases} 3x + 2y > 37. & (1) \\ 2x + 3y = 33. & (2) \end{cases}$$

Multiply (1) by 3, $9x + 6y > 111.$
Multiply (2) by 2, $4x + 6y = 66.$
Subtracting (§ 56); $5x > 45$, and $x > 9$.

Multiply (1) by 2, $6x + 4y > 74.$
Multiply (2) by 3, $6x + 9y = 99.$
Subtracting, $-5y > -25$

Divide both members by -5, $y < 5$ (§ 60).

(This means that any values of x and y which satisfy (2), also satisfy (1), provided x is > 9, and $y < 5$.)

3. Between what limiting values of x is $x^2 - 4x < 21$?

Transposing 21, we have

$$x^2 - 4x < 21, \text{ if } x^2 - 4x - 21 < 0.$$

That is, if $(x + 3)(x - 7)$ is negative.

Now $(x + 3)(x - 7)$ is negative if x is between -3 and 7; for if $x < -3$, both $x + 3$ and $x - 7$ are negative, and their product positive; and if $x > 7$, both $x + 3$ and $x - 7$ are positive.

Hence, $x^2 - 4x < 21$, if $x > -3$, and < 7.

EXERCISE 6

Find the limits of x in the following:

1. $(4x + 5)^2 - 4 < (8x + 5)(2x + 3)$.
2. $(3x + 2)(x + 3) - 4x > (3x - 2)(x - 3) + 36$.
3. $(x + 4)(5x - 2) + (2x - 3)^2 > (3x + 4)^2 - 78$.

4. $(x-3)(x+4)(x-5)<(x+1)(x-2)(x-3)$.

5. $a^2(x-1)<2b^2(2x-1)-ab$, if $a-2b$ is positive.

Find the limits of x and y in the following:

6. $\begin{cases} 5x+6y<45. \\ 3x-4y=-11. \end{cases}$

7. $\begin{cases} 7x-4y>41. \\ 3x+7y=35. \end{cases}$

8. Find the limits of x when

$$3x-11<24-11x, \text{ and } 5x+23<20x+3.$$

9. If 6 times a certain positive integer, plus 14, is greater than 13 times the integer, minus 63, and 17 times the integer, minus 23, is greater than 8 times the integer, plus 31, what is the integer?

10. If 7 times the number of houses in a certain village, plus 33, is less than 12 times the number, minus 82, and 9 times the number, minus 43, is less than 5 times the number, plus 61, how many houses are there?

11. A farmer has a number of cows such that 10 times their number, plus 3, is less than 4 times the number, plus 79; and 14 times their number, minus 97, is greater than 6 times the number, minus 5. How many cows has he?

12. Between what limiting values of x is $x^2+3x<4$?

13. Between what limiting values of x is $x^2<8x-15$?

14. Between what limiting values of x is $3x^2+19x<-20$?

65. If a and b are unequal numbers,

$$a^2+b^2>2ab.$$

For $\qquad (a-b)^2>0$; or, $a^2-2ab+b^2>0$.

Transposing $-2ab$, $\qquad a^2+b^2>2ab$.

1. Prove that, if a does not equal 3,

$$(a+2)(a-2)>6a-13.$$

By the above principle, if a does not equal 3,

$$a^2 + 9 > 6a.$$

Subtracting 13 from both members,

$$a^2 - 4 > 6a - 13, \text{ or } (a+2)(a-2) > 6a - 13.$$

2. Prove that, if a and b are unequal positive numbers,

$$a^3 + b^3 > a^2b + b^2a.$$

We have, $a^2 + b^2 > 2ab$, or $a^2 - ab + b^2 > ab$.

Multiplying both members by the positive number $a + b$,

$$a^3 + b^3 > a^2b + b^2a.$$

EXERCISE 7

1. Prove that for any value of x, except $\frac{5}{3}$,

$$3x(3x - 10) > -25.$$

2. Prove that for any value of x, except $\frac{7}{2}$,

$$4x(x-5) > 8x - 49.$$

3. Prove that for any values of a and b, if $4a$ does not equal $3b$, $$(4a + 3b)(4a - 3b) > 6b(4a - 3b).$$

4. Prove that for any values of x and y, if $5x$ does not equal $4y$, $$5x(5x - 6y) > 2y(5x - 8y).$$

Prove that, if a and b are unequal positive numbers,

5. $a^3b + ab^3 > 2a^2b^2$.

6. $a^3 + a^2b + ab^2 + b^3 > 2ab(a+b)$.

III. EXPONENTS

66. An **Exponent** is a number written at the right of and above a number.

It is customary to speak of the number as raised to the power indicated by the exponent.

67. The laws we shall develop are to hold for any exponent, whether integral, fractional, positive, negative, or zero.

68. The number raised to the power is called the **Base**.

69. Meaning of a Positive Integral Exponent.

$$a^3 = a \cdot a \cdot a.$$

$$a^4 = a \cdot a \cdot a \cdot a.$$

Similarly if m is a positive integer,

$$a^m = a \cdot a \cdot a \cdot \cdots \text{ to } m \text{ factors.}$$

The following results have been proved to hold for any positive integral values of m and n:

$$a^m \times a^n = a^{m+n} \text{ (F. C.)*} \tag{1}$$

$$(a^m)^n = a^{mn} \text{ (F. C.)} \tag{2}$$

70. Meaning of a Fractional Exponent.

Let it be required to find the meaning of $a^{\frac{5}{3}}$.

If (1), § 69, is to hold for all values of m and n,

$$a^{\frac{5}{3}} \times a^{\frac{5}{3}} \times a^{\frac{5}{3}} = a^{\frac{5}{3}+\frac{5}{3}+\frac{5}{3}} = a^5.$$

Then, the *third power* of $a^{\frac{5}{3}}$ equals a^5.

Hence, $a^{\frac{5}{3}}$ must be the *cube root* of a^5, or $a^{\frac{5}{3}} = \sqrt[3]{a^5}$.

We will now consider the general case.

Let it be required to find the meaning of $a^{\frac{p}{q}}$, where p and q are any positive integers.

* F. C. refers to Wells's First Course in Algebra.

If (1), § 69, is to hold for all values of m and n,

$$a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times \cdots \text{ to } q \text{ factors} = a^{\frac{p}{q}+\frac{p}{q}+\frac{p}{q}+\cdots \text{ to } q \text{ terms}} = a^{\frac{p}{q}\times q} = a^p.$$

Then, the qth power of $a^{\frac{p}{q}}$ equals a^p.

Hence, $a^{\frac{p}{q}}$ must be the qth root of a^p, or $a^{\frac{p}{q}} = \sqrt[q]{a^p}$.

Hence, in a fractional exponent, the numerator denotes a power, and the denominator a root.

For example, $a^{\frac{3}{4}} = \sqrt[4]{a^3}$; $b^{\frac{5}{2}} = \sqrt{b^5}$; $x^{\frac{1}{3}} = \sqrt[3]{x}$; etc.

A **Surd** is the indicated root of a number, or expression, which is not a perfect power of the degree denoted by the index of the radical sign; as $\sqrt{2}$, $\sqrt[3]{5}$, or $\sqrt[4]{x+y}$.

The **degree** of a surd is denoted by its index; thus, $\sqrt[3]{5}$ is a surd of the third degree.

A **quadratic surd** is a surd of the second degree.

71. Meaning of a Zero Exponent.

If (1), § 69, is to hold for all values of m and n, we have

$$a^m \times a^0 = a^{m+0} = a^m.$$

Whence,
$$a^0 = \frac{a^m}{a^m} = 1.$$

We must then define a^0 as being equal to 1.

72. Meaning of a Negative Exponent.

Let it be required to find the meaning of a^{-3}.

If (1), § 69, is to hold for all values of m and n,

$$a^{-3} \times a^3 = a^{-3+3} = a^0 = 1 \text{ (§ 71)}.$$

Whence,
$$a^{-3} = \frac{1}{a^3}.$$

We will now consider the general case.

Let it be required to find the meaning of a^{-s}, where s represents a positive integer or a positive fraction.

If (1), § 69, is to hold for all values of m and n,

$$a^{-s} \times a^{s} = a^{-s+s} = a^{0} = 1 \text{ (§ 71).}$$

Whence, $$a^{-s} = \frac{1}{a^{s}}.$$

We must then define a^{-s} as being equal to 1 divided by a^{s}.

For example, $a^{-2} = \frac{1}{a^{2}}$; $a^{-\frac{2}{3}} = \frac{1}{a^{\frac{2}{3}}}$; $3\,x^{-1}y^{-\frac{1}{2}} = \frac{3}{xy^{\frac{1}{2}}}$; etc.,

73. It follows from § 72 that

Any factor of the numerator of a fraction may be transferred to the denominator, or any factor of the denominator to the numerator, if the sign of its exponent be changed.

Thus, $$\frac{a^2b^3}{cd^4} = \frac{b^3}{a^{-2}cd^4} = \frac{a^2b^3c^{-1}}{d^4} = \frac{a^2d^{-4}}{b^{-3}c}, \text{ etc.}$$

EXERCISE 8

Express with positive exponents:

1. $a^{-2}b^3$.
2. $x^{\frac{3}{4}}y^{-2}z$.
3. $2\,m^{-4}n$.
4. $a^{-1}b^4c^{-3}$.
5. $3\,xyz^{-2}$.
6. $5\,c^{-\frac{1}{2}}d^{\frac{1}{3}}$.
7. $a^{-2}xy^{-5}$.
8. $3\,p^{-1}q^{\frac{1}{2}}$.
9. $7\,x^4y^{-2}z$.
10. $4\,a^{-6}b^{-8}c^{\frac{3}{2}}$.
11. $8\,u^3v^{-1}$.
12. $r^{\frac{1}{4}}s^{-\frac{1}{2}}t^{-\frac{1}{3}}$.

Transfer all literal factors from the denominators to the numerators:

13. $\frac{6\,x^3}{y}$.
14. $\frac{mn^{-4}}{3\,x^2}$.
15. $\frac{abc^{-1}}{xy^2z}$.
16. $\frac{1}{2\,a^2b^{-3}}$.
17. $\frac{a^4}{cd^{-3}}$.
18. $\frac{3\,x^2y^{-\frac{1}{4}}}{4\,z^4}$.
19. $\frac{5\,a^{\frac{1}{2}}b^{-\frac{1}{3}}}{9\,c^{-3}d^{\frac{3}{4}}}$.
20. $\frac{4\,m^{-2}n^3}{7\,x^{-8}y^{-\frac{5}{2}}z}$.

Transfer all literal factors from the numerators to the denominators:

21. $\dfrac{a^3b^2}{x^4}$.

22. $\dfrac{7\,x^2y^{-1}}{m^3n^{\frac{3}{2}}}$.

23. $\dfrac{3\,ab^2c^{-1}}{5\,d^{-4}}$.

24. $\dfrac{p^{-8}q^{\frac{3}{5}}}{5\,x^3}$.

25. $\dfrac{m^6}{n^{-7}r^{\frac{3}{2}}}$.

26. $\dfrac{8\,x^{-4}yz^3}{3\,cd^2}$.

27. $\dfrac{a^{-2}}{b^2c}$.

28. $\dfrac{3\,m^{\frac{1}{4}}n^{-\frac{2}{3}}}{4\,xy^{-1}z^2}$.

74. Proof that $a^m \cdot a^n = a^{m+n}$ holds for all values of m and n.

I. Let $m = \dfrac{p}{q}$ and $n = \dfrac{r}{s}$, where p, q, r, and s are positive integers.

We have, $$a^{\frac{p}{q}} \times a^{\frac{r}{s}} = a^{\frac{ps}{qs}} \times a^{\frac{qr}{qs}} = \sqrt[qs]{a^{ps}} \times \sqrt[qs]{a^{qr}} \text{ (§ 70)}$$
$$= \sqrt[qs]{a^{ps} \times a^{qr}} = \sqrt[qs]{a^{ps+qr}} = a^{\frac{ps+qr}{qs}} \text{ (§ 70)} = a^{\frac{p}{q}+\frac{r}{s}}.$$

We have now proved that (1), § 69, holds when m and n are any positive integers or positive fractions.

II. Let m be a positive integer or fraction; and let $n = -q$, where q is a positive integer or fraction less than m.

By § 74, I, $$a^{m-q} \times a^q = a^{m-q+q} = a^m.$$

Whence, $$a^{m-q} = \frac{a^m}{a^q} = a^m \times a^{-q} \text{ (§ 73)}.$$

That is, $$a^m \times a^{-q} = a^{m-q}.$$

III. Let m be a positive integer or fraction; and let $n = q$, where q is a positive integer or fraction greater than m.

By § 73, $$a^m \times a^{-q} = \frac{1}{a^{-m}a^q} = \frac{1}{a^{-m+q}} \text{ (§ 74, II)} = a^{m-q}.$$

IV. Let $m = -p$ and $n = -q$, where p and q are positive integers or fractions.

Then, $$a^{-p} \times a^{-q} = \frac{1}{a^pa^q} = \frac{1}{a^{p+q}} \text{ (74, I)} = a^{-p-q}.$$

Then, $a^m \times a^n = a^{m+n}$ for all positive or negative, integral or fractional, values of m and n.

EXERCISE 9

Multiply the following:

1. a^3 by $a^{-\frac{3}{2}}$.
2. $3x^{\frac{1}{2}}y^{-\frac{1}{3}}z$ by $x^{-\frac{3}{4}}yz^3$.
3. $2c^{\frac{1}{4}}d$ by $3\sqrt{cd}^{\frac{1}{2}}$.
4. $2\sqrt[4]{ab^{-3}}$ by $\sqrt[3]{a^2b}$.
5. $x^{\frac{3}{4}}y$ by $\dfrac{1}{x^{-\frac{5}{6}}y^{-2}}$.
6. a^4bc^2 by $ab^{-1}c^{-2}$.
7. $3x^{-4}y^0$ by $-2x^6y^3z$.
8. $x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$ by $x^{\frac{1}{3}}-y^{\frac{1}{3}}$.
9. $3x-1+x^{-1}$ by $5x+2$.
10. $x^{\frac{1}{2}}-2x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}}$ by $\sqrt{x}+\sqrt{y}$.
11. $x^2+2x-x^{-1}+1$ by $x+x^{-1}+1$.
12. $\dfrac{1}{x^{-2}}-x^{\frac{3}{2}}+x-\sqrt{x}+1$ by $\sqrt{x}+1$.

Divide the following:

13. a^2 by a^5.
14. $x^{\frac{1}{4}}y$ by $x^{-3}y^2$.
15. $3\sqrt{xy^3}$ by $x^{-\frac{1}{3}}y^{-1}$.
16. $8a^{\frac{1}{2}}b^{-\frac{1}{3}}\sqrt[3]{c^2}$ by $2a^{-2}bc^3$.
17. $5m^4n^{-\frac{1}{6}}$ by $2m^{-\frac{3}{2}}n^{\frac{1}{3}}$.
18. $a+2a^{\frac{1}{2}}b^{\frac{1}{2}}+b$ by $a^{\frac{1}{2}}+b^{\frac{1}{2}}$.
19. $m^{-\frac{3}{2}}+3m^{-1}n^{\frac{1}{2}}+3m^{-\frac{1}{2}}n+n^{\frac{3}{2}}$ by $m^{-\frac{1}{2}}+n^{\frac{1}{2}}$.
20. $6x^2-6x^{-2}-12-11x+23x^{-1}$ by $2x+1-3x^{-1}$.
21. $9x^{\frac{1}{3}}y^{-1}-6x^{\frac{2}{3}}y^{-2}-5+12x^{-\frac{1}{3}}y+4x^{-\frac{2}{3}}y^2$ by $3x+4\sqrt[3]{xy^2}$.
22. $2x^{\frac{3}{4}}-7x^{\frac{1}{2}}+4x^{-\frac{1}{4}}-11+12x^{\frac{1}{4}}$ by $2x^{\frac{1}{2}}+4-3x^{\frac{1}{4}}$.

Find the values of the following:

23. $(x^2)^{-3}$.
24. $(m^{-2})^{-\frac{3}{4}}$.
25. $(\sqrt[3]{x^{-2}})^{-\frac{5}{6}}$.
26. $(36)^{\frac{3}{2}}$.
27. $(-27)^{\frac{5}{3}}$.
28. $(8^{-\frac{1}{3}})^4$.
29. $\left(x^{\frac{m}{m+n}}\right)^{1-\frac{n^2}{m^2}}$.
30. $(\sqrt{a^q})^{-\frac{1}{q}}$.

75. To prove the result

$$(ab)^n = a^n b^n,$$

for any fractional or negative value of n.

The proof of this result in the case where n is any positive integer, was given in F. C.

I. Let $n = \frac{p}{q}$, where p and q are any positive integers.

$$[(ab)^{\frac{p}{q}}]^q = (ab)^p = a^p b^p \text{ (F. C.)}. \qquad (1)$$

$$(a^{\frac{p}{q}} b^{\frac{p}{q}})^q = (a^{\frac{p}{q}})^q (b^{\frac{p}{q}})^q = a^p b^p. \qquad (2)$$

From (1) and (2), $[(ab)^{\frac{p}{q}}]^q = (a^{\frac{p}{q}} b^{\frac{p}{q}})^q$.

Taking the qth root of both members, we have

$$(ab)^{\frac{p}{q}} = a^{\frac{p}{q}} b^{\frac{p}{q}}.$$

II. Let $n = -s$, where s is any positive integer or positive fraction.

Then, $(ab)^{-s} = \frac{1}{(ab)^s} = \frac{1}{a^s b^s}$ (§ 74, IV) $= a^{-s} b^{-s}$.

EXERCISE 10

Find the values of the following:

1. $(a^3 b^{\frac{1}{2}})^4$.
2. $\left(\frac{1}{x^{\frac{1}{2}} y^{-3}}\right)^{-\frac{2}{3}}$.
3. $(p^{-\frac{2}{3}} q^{\frac{3}{4}})^{-\frac{5}{6}}$.
4. $(a^{-4} \sqrt{b^3} c^{-1})^{-\frac{1}{2}}$.
5. $(\sqrt[3]{x^4 y^{-3}})^{\frac{7}{8}}$.
6. $(25\, a^4)^{-\frac{5}{2}}$.
7. $(32\, a^{\frac{1}{4}} \sqrt[3]{b^{-2}})^{\frac{3}{5}}$.
8. $(343\, b^4 c^{-1} d^{\frac{1}{2}})^{-\frac{2}{3}}$.
9. $\left(\frac{9\, x y^{-2} z^{\frac{1}{3}}}{16\, m^{-4}}\right)^{-\frac{3}{2}}$.
10. $\left(\frac{3\, a^{-4} b^{\frac{1}{3}} c}{4\, x^{-\frac{2}{3}} y^{-\frac{1}{3}}}\right)^{-2}$.

EXERCISE 11

Illustrative Examples.

Ex. 1. Reduce $\sqrt{\frac{7}{8}}$ to its simplest form.

A surd is said to be in its *simplest form* when the expression under the radical sign is rational and integral, is not a perfect power of the degree denoted by any factor of the index of the surd, and has no factor which is a perfect power of the same degree as the surd.

$8 = 2^3$. To be a perfect square the exponents of the factors of the denominator must be even numbers. Hence multiplying both terms of the fraction by 2, we have,

$$\sqrt{\frac{7}{8}} = \sqrt{\frac{14}{16}} = \frac{\sqrt{14}}{4}.$$

Ex. 2. Reduce $\sqrt[4]{25}$ to its simplest form.

$$\sqrt[4]{25} = \sqrt{\sqrt{25}} = \sqrt{5}.$$

Ex. 3. Express $5\sqrt{7}$ entirely under the radical sign.

$$5\sqrt{7} = \sqrt{5^2}\sqrt{7}, \text{ or } (5^2)^{\frac{1}{2}}(7)^{\frac{1}{2}}.$$

By § 75, $\quad (5^2)^{\frac{1}{2}}(7)^{\frac{1}{2}} = (5^2 \cdot 7)^{\frac{1}{2}} = \sqrt{175}.$

Ex. 4. Reduce $(5)^{\frac{1}{3}}$, $\sqrt[4]{3}$ to the same degree.

The L. C. M. of the indices of the roots is 12. Hence,

$$5^{\frac{1}{3}} = 5^{\frac{4}{12}}, \quad 3^{\frac{1}{4}} = 3^{\frac{3}{12}}.$$

The surds are now of the twelfth degree.

Ex. 5. Find the product of $\sqrt{45}$ and $\sqrt{72}$.

$$(45)^{\frac{1}{2}}(72)^{\frac{1}{2}} = (3^2 \cdot 5)^{\frac{1}{2}} \cdot (3^2 \cdot 2^3)^{\frac{1}{2}} = (2^2 \cdot 3^4 \cdot 5 \cdot 2)^{\frac{1}{2}}$$
$$= 2 \cdot 3^2 (5 \cdot 2)^{\frac{1}{2}} = 18\sqrt{10}.$$

Reduce the following to their simplest form:

1. $\sqrt[4]{9}$.
2. $(27\,a^3)^{\frac{1}{6}}$.
3. $(45)^{\frac{1}{2}}$.
4. $(72)^{\frac{1}{2}}$.
5. $\sqrt[3]{128\,a^4b^2}$.
6. $3\sqrt{250\,a^2x}$.
7. $5(32\,a^5x^4y^3)^{\frac{1}{4}}$.
8. $7(80)^{\frac{1}{2}}$.
9. $(686)^{\frac{1}{3}}$.

10. $4\sqrt[5]{486}.$

11. $\sqrt{4x^2 - 5x^3y}.$

12. $(a^3 - 2a^2b + ab^2)^{\frac{1}{2}}.$

13. $(x-y)(ax^3m^5)^{\frac{1}{2}}.$

14. $(4x^2 - 24xb + 36b^2)^{\frac{1}{2}}.$

15. $\sqrt{(x^2+3x+2)(x^2+6x+8)}$

16. $(\frac{1}{3})^{\frac{1}{2}}.$

17. $(\frac{3}{8})^{\frac{1}{2}}.$

18. $(\frac{1}{4})^{\frac{1}{3}}.$

19. $\frac{1}{2}(\frac{1}{5})^{\frac{1}{2}}.$

20. $2\sqrt{\frac{5}{72}}.$

21. $\left(\frac{8a^3}{75}\right)^{\frac{1}{2}}.$

22. $\frac{1}{m}\left(\frac{18m^4}{25n^2}\right)^{\frac{1}{3}}.$

23. $(x-a)\left(\frac{1}{x+a}\right)^{\frac{1}{2}}.$

24. $\sqrt{\frac{3p}{p-2q}}.$

25. $\frac{1}{(a+b)^2}\left(\frac{a^2-b^2}{6}\right)^{\frac{1}{2}}.$

Express entirely under the radical sign:

26. $2\sqrt{5}.$

27. $3(2)^{\frac{1}{3}}.$

28. $a(bc^2)^{\frac{1}{4}}.$

29. $5x\sqrt{3xy}.$

30. $(a+3b)\left(\frac{1}{a+3b}\right)^{\frac{1}{2}}.$

31. $(x+y)\left(\frac{x}{x^2-y^2}\right)^{\frac{1}{3}}.$

32. $\frac{1}{m-n}\sqrt{m^2+mn-2n^2}.$

33. $\frac{c+4}{c-1}\left(\frac{c^2+5c-6}{c^2+8c+16}\right)^{\frac{1}{2}}.$

Reduce the following to equivalent surds of the same degree:

34. $\sqrt{3}, \sqrt[3]{4}.$

35. $(2)^{\frac{1}{2}}, (\frac{1}{4})^{\frac{1}{3}}, (5)^{\frac{1}{4}}.$

36. $(x^2y)^{\frac{1}{3}}, (xy)^{\frac{1}{2}}, (x^3y^2)^{\frac{1}{4}}.$

37. $5\sqrt[3]{x}, 3\sqrt{xy}.$

38. $(a-x)^{\frac{1}{2}}, (a+x)^{\frac{1}{3}}, (a^2+x^2)^{\frac{1}{6}}.$

39. $\sqrt[6]{x^3+y^3}, \sqrt[9]{x^4-y^4}.$

Simplify the following:

40. $(18)^{\frac{1}{2}} + 3(50)^{\frac{1}{2}} - 2(72)^{\frac{1}{2}}.$

41. $2(27)^{\frac{1}{2}} - 5(48)^{\frac{1}{2}} + 11(75)^{\frac{1}{2}}.$

42. $\sqrt[3]{54} + \sqrt[3]{250} - 2\sqrt[3]{128}.$

43. $\frac{5}{2}(12)^{\frac{1}{2}} - \frac{3}{4}(\frac{1}{3})^{\frac{1}{2}} + (3)^{\frac{1}{2}}.$

44. $8\sqrt[4]{80} - 2\sqrt[4]{405} + 18\sqrt[4]{\frac{5}{16}}.$

45. $(24\,a^3x)^{\frac{1}{2}} + 2(54\,a^3x^3)^{\frac{1}{2}} - 5(6\,a^5x)^{\frac{1}{2}}$.

46. $\left(\dfrac{a^3m^5}{c^4n}\right)^{\frac{1}{2}} - \left(\dfrac{amn^3}{c^2d^2}\right)^{\frac{1}{2}} + \left(\dfrac{a^5n^3}{m^3x^4}\right)^{\frac{1}{2}}$.

47. $(\frac{7}{2})^{\frac{1}{2}} + 3(\frac{1}{14})^{\frac{1}{2}} + \frac{1}{2}\sqrt{56}$.

48. $\sqrt[3]{2\,ax} - 3(4\,a^2x^2)^{\frac{1}{6}} + 5\sqrt[9]{8\,a^3x^3}$.

49. $\dfrac{1}{(x+y)^2}\sqrt{x^3 + 2\,x^2y + xy^2} + \dfrac{x}{x^2 - y^2}\left(\dfrac{(x+y)^4}{x}\right)^{\frac{1}{2}}$.

Multiply the following:

50. $\sqrt{90}$ by $\sqrt{63}$.

51. $(35)^{\frac{1}{2}}$ by $(105)^{\frac{1}{2}}$.

52. $\sqrt[3]{54}$ by $\sqrt[3]{6}$.

53. $(3)^{\frac{1}{2}}$ by $(2)^{\frac{1}{3}}$.

54. $\sqrt[3]{2}$ by $\sqrt[4]{4}$.

55. $(\frac{1}{2})^{\frac{1}{2}}$ by $(\frac{3}{4})^{\frac{1}{2}}$ by $(\frac{2}{3})^{\frac{1}{2}}$.

56. $\sqrt{a^3xy}$ by $\sqrt[3]{ax^2y^2}$.

57. $2(5)^{\frac{1}{2}}$ by $3(15)^{\frac{1}{2}}$.

58. $5\sqrt[3]{40}$ by $6(5)^{\frac{1}{2}}$.

59. $\left(\dfrac{3\,a}{8\,b}\right)^{\frac{1}{2}} \cdot \left(\dfrac{7\,a^2}{27\,b}\right)^{\frac{1}{2}} \cdot \left(\dfrac{2\,b^3}{15\,a}\right)^{\frac{1}{2}}$.

60. $\sqrt{3\,ax} \cdot (2\,a^2)^{\frac{1}{3}} \cdot (6\,x^3)^{\frac{1}{4}}$.

61. $3(2)^{\frac{1}{2}} - 5(3)^{\frac{1}{2}}$ by $4(2)^{\frac{1}{2}} + 3(3)^{\frac{1}{2}}$.

62. $5\sqrt{7} + 6\sqrt{2}$ by $\sqrt{7} - 4\sqrt{2}$.

63. $2(8\,x)^{\frac{1}{2}} - 9(2\,y)^{\frac{1}{2}}$ by $(2\,x)^{\frac{1}{2}} - 3(2\,y)^{\frac{1}{2}}$.

64. $3(a-1)^{\frac{1}{2}} + 4(2\,a+5)^{\frac{1}{2}}$ by $2(a-1)^{\frac{1}{2}} - 10(2\,a+5)^{\frac{1}{2}}$.

65. $5\sqrt{\frac{3}{5}} - 2\sqrt{\frac{1}{3}}$ by $4\sqrt{\frac{3}{5}} + 9\sqrt{\frac{1}{3}}$.

Divide the following:

66. $\sqrt{72}$ by $\sqrt{6}$.

67. $2\sqrt{125}$ by $4\sqrt{5}$.

68. $(192)^{\frac{1}{3}}$ by $(12)^{\frac{1}{3}}$.

69. $(512)^{\frac{1}{3}}$ by $(16)^{\frac{1}{2}}$.

70. $(8\,a^3)^{\frac{1}{2}}$ by $(16\,a^4)^{\frac{1}{3}}$.

71. $\sqrt{722\,x^5}$ by $\sqrt{2\,a^4x}$.

72. $(\frac{1}{2})^{\frac{1}{3}}$ by $(\frac{1}{2})^{\frac{1}{2}}$.

73. $8\sqrt{a^3x^2}$ by $6\sqrt[3]{ax^2}$.

74. $(1\frac{1}{3})^{\frac{1}{2}}$ by $(9)^{\frac{1}{3}}$.

Find values of the following:

75. $(a\sqrt{5})^3$.

76. $[\frac{1}{2}(4)^{\frac{1}{3}}]^2$.

77. $(3\sqrt[5]{2a^2b})^4$.

78. $[(3)^{\frac{1}{3}}]^{\frac{1}{2}}$.

79. $[(3a)^{\frac{1}{2}}]^{\frac{1}{3}}$.

80. $\sqrt{\sqrt[3]{a^2 - 2ab + b^2}}$.

81. $[3 + 2(5)^{\frac{1}{2}}]^2$.

82. $(4\sqrt{5} - 2\sqrt{7})^2$.

83. $[3(3)^{\frac{1}{4}} - 5(2)^{\frac{1}{2}}][3(3)^{\frac{1}{2}} + 3(2)^{\frac{1}{2}}]$.

84. $(7\sqrt{11} - 5\sqrt{3})(7\sqrt{11} + 5\sqrt{3})$.

Express each of the following with a rational denominator:

85. $\dfrac{1}{\sqrt{2}}$.

86. $\dfrac{2a}{\sqrt{5}}$.

87. $\dfrac{x}{(x)^{\frac{1}{2}} + (a)^{\frac{1}{2}}}$.

88. $\dfrac{a + b^{\frac{1}{2}}}{a - b^{\frac{1}{2}}}$.

89. $\dfrac{2 + \sqrt{3}}{2 - \sqrt{3}}$.

90. $\dfrac{3(4)^{\frac{1}{2}} - 2(3)^{\frac{1}{2}}}{2(3)^{\frac{1}{2}} + (10)^{\frac{1}{2}}}$.

91. $\dfrac{(a+b)^{\frac{1}{2}} + (a-b)^{\frac{1}{2}}}{(a+b)^{\frac{1}{2}} - (a-b)^{\frac{1}{2}}}$.

92. $\dfrac{(x^2+y^2)^{\frac{1}{2}} + (x^2-y^2)^{\frac{1}{2}}}{(x^2+y^2)^{\frac{1}{2}} - (x^2-y^2)^{\frac{1}{2}}}$.

LOGARITHMS

76. Any number may be expressed as a power of some number chosen as base.

For example, $4 = 2^2$, $8 = 2^3$, $64 = 2^6$, etc. Numbers between 4 and 8 would be expressed by 2^n where n is 2 plus some fractional number. In such a case the exponent is called the **Logarithm of the Number to the Base 2.**

E.g. 2 is the logarithm of 4 to the base 2; 3 is the logarithm of 8 to the base 2, etc.

77. The **Common System** of logarithms has 10 for its base.

Every positive arithmetical number may be expressed, exactly or approximately, as a power of 10.

Thus, $100 = 10^2$; $13 = 10^{1.1139\cdots}$; etc.

When thus expressed, the corresponding exponent is called its **Logarithm to the Base 10.**

Thus, 2 is the logarithm of 100 to the base 10; a relation which is written $\log_{10} 100 = 2$, or simply $\log 100 = 2$.

Logarithms of numbers to the base 10 are called *Common Logarithms,* and, collectively, form the *Common System.*

They are the only ones used for numerical computations.

78. Any positive number, except unity, may be taken as the base of a system of logarithms; thus, if $a^x = m$, where a and m are positive numbers, then $x = \log_a m$.

A negative number is not considered as having a logarithm.

79. By §§ 71 and 72,

$$10^0 = 1, \qquad 10^{-1} = \frac{1}{10} = .1,$$

$$10^1 = 10, \qquad 10^{-2} = \frac{1}{10^2} = .01,$$

$$10^2 = 100, \qquad 10^{-3} = \frac{1}{10^3} = .001, \text{ etc.}$$

Whence, by the definition of § 76,

$$\log 1 = 0, \qquad \log .1 = -1 = 9 - 10,$$

$$\log 10 = 1, \qquad \log .01 = -2 = 8 - 10,$$

$$\log 100 = 2, \qquad \log .001 = -3 = 7 - 10, \text{ etc.}$$

The second form for log .1, log .01, etc., is preferable in practice.

If no base is expressed, the base 10 is understood.

80. It is evident from § 79 that the common logarithm of a number greater than 1 is positive, and the logarithm of a number between 0 and 1 negative.

81. If a number is not an exact power of 10, its common logarithm can only be expressed approximately; the integral part of the logarithm is called the *characteristic,* and the decimal part the *mantissa.*

For example, $\log 13 = 1.1139.$

Here, the characteristic is 1, and the mantissa .1139.

A negative logarithm is always expressed with a positive mantissa, which is done by adding and subtracting 10.

Thus, the negative logarithm -2.5863 is written $7.4137-10$. In this case, $7-10$ is the characteristic.

The negative logarithm $7.4137-10$ is sometimes written $\bar{3}.4137$; the negative sign over the characteristic showing that it alone is negative, the mantissa being always positive.

For reasons which will appear, only the mantissa of the logarithm is given in a table of logarithms of number; the characteristic must be found by aid of the rules of §§ 82 and 83.

82. It is evident from § 79 that the logarithm of a number between

1 and 10 is equal to 0 + a decimal;
10 and 100 is equal to 1 + a decimal;
100 and 1000 is equal to 2 + a decimal; etc.

Therefore, the characteristic of the logarithm of a number with *one* place to the left of the decimal point is 0; with *two* places to the left of the decimal point is 1; with *three* places to the left of the decimal point is 2; etc.

Hence, **the characteristic of the logarithm of a number greater than 1 is 1 less than the number of places to the left of the decimal point.**

For example, the characteristic of log 906328.51 is 5.

83. In like manner, the logarithm of a number between

1 and .1 is equal to 9 + a decimal − 10;
.1 and .01 is equal to 8 + a decimal − 10;
.01 and .001 is equal to 7 + a decimal − 10; etc.

Therefore, the characteristic of the logarithm of a decimal with *no* ciphers between its decimal point and first significant figure is 9, with −10 after the mantissa; of a decimal with *one* cipher between its point and first significant figure is 8, with −10 after the mantissa; of a decimal with *two* ciphers between its point and first significant figure is 7, with −10 after the mantissa; etc.

Hence, to find the characteristic of the logarithm of a number less than 1, subtract the number of ciphers between the decimal point and first significant figure from 9, writing − 10 after the mantissa.

For example, the characteristic of log .007023 is 7, with − 10 written after the mantissa.

PROPERTIES OF LOGARITHMS

84. *In any system, the logarithm of* 1 *is* 0.

For by § 71, $a^0 = 1$; whence, by § 78, $\log_a 1 = 0$.

85. *In any system the logarithm of the base is* 1.

For, $a^1 = a$; whence, $\log_a a = 1$.

86. *In any system whose base is greater than* 1, *the logarithm of* 0 *is* $-\infty$.*

For if a is greater than 1, $a^{-\infty} = \frac{1}{a^\infty} = \frac{1}{\infty} = 0$. (The discussion of this form will be found in § 127.)

Whence, by § 78, $\log_a 0 = -\infty$.

No literal meaning can be attached to such a result as $\log_a 0 = -\infty$; it must be interpreted as follows:

If, in any system whose base is greater than unity, a number approaches the limit 0, its logarithm is negative, and increases indefinitely in absolute value.

87. *In any system, the logarithm of a product is equal to the sum of the logarithms of its factors.*

Assume the equations

$$\left.\begin{matrix} a^x = m \\ a^y = n \end{matrix}\right\}; \text{ whence, by § 78, } \left\{\begin{matrix} x = \log_a m, \\ y = \log_a n. \end{matrix}\right.$$

Multiplying the assumed equations,

$$a^x \times a^y = mn, \text{ or } a^{x+y} = mn.$$

Whence, $$\log_a mn = x + y = \log_a m + \log_a n.$$

* ∞ stands for a number greater than any assigned number. See § 126.

In like manner, the theorem may be proved for the product of three or more factors.

By aid of § 87, the logarithm of a composite number may be found when the logarithms of its factors are known.

Ex. Given $\log 2 = .3010$, and $\log 3 = .4771$; find $\log 72$.

$$\begin{aligned}\log 72 &= \log(2 \times 2 \times 2 \times 3 \times 3)\\ &= \log 2 + \log 2 + \log 2 + \log 3 + \log 3\\ &= 3 \times \log 2 + 2 \times \log 3 = .9030 + .9542 = 1.8572.\end{aligned}$$

EXERCISE 12

Given $\log 2 = .3010$, $\log 3 = .4771$, $\log 5 = .6990$, $\log 7 = .8451$, find:

1. $\log 15$.	4. $\log 125$.	7. $\log 567$.	10. $\log 1875$.
2. $\log 98$.	5. $\log 315$.	8. $\log 1225$.	11. $\log 2646$.
3. $\log 84$.	6. $\log 392$.	9. $\log 1372$.	12. $\log 24696$.

88. In any system, the logarithm of a fraction is equal to the logarithm of the numerator minus the logarithm of the denominator.

Assume the equations

$$\left.\begin{array}{l} a^x = m \\ a^y = n \end{array}\right\}; \text{ whence, } \left\{\begin{array}{l} x = \log_a m, \\ y = \log_a n. \end{array}\right.$$

Dividing the assumed equations,

$$\frac{a^x}{a^y} = \frac{m}{n}, \text{ or } a^{x-y} = \frac{m}{n}.$$

Whence, $$\log_a \frac{m}{n} = x - y = \log_a m - \log_a n.$$

Ex. Given $\log 2 = .3010$; find $\log 5$.

$$\log 5 = \log \frac{10}{2} = \log 10 - \log 2 = 1 - .3010 = .6990.$$

EXERCISE 13

Given $\log 2 = .3010$, $\log 3 = .4771$, $\log 7 = .8451$, find:

1. $\log \frac{10}{7}$.	4. $\log 245$.	7. $\log \frac{49}{48}$.	10. $\log \frac{300}{49}$.
2. $\log \frac{27}{2}$.	5. $\log 85\frac{3}{4}$.	8. $\log 375$.	11. $\log 46\frac{2}{7}$.
3. $\log 11\frac{1}{9}$.	6. $\log 175$.	9. $\log \frac{54}{25}$.	12. $\log 2\frac{11}{35}$.

89. In any system, the logarithm of any power of a number is equal to the logarithm of the number multiplied by the exponent of the power.

Assume the equation $a^x = m$; whence, $x = \log_a m$.

Raising both members of the assumed equation to the pth power, $a^{px} = m^p$; whence, $\log_a m^p = px = p \log_a m$.

90. In any system, the logarithm of any root of a number is equal to the logarithm of the number divided by the index of the root.

For, $\log_a \sqrt[r]{m} = \log_a(m^{\frac{1}{r}}) = \frac{1}{r} \log_a m$ (§ 89).

91. Examples.

1. Given $\log 2 = .3010$; find $\log 2^{\frac{5}{3}}$.

$$\log 2^{\frac{5}{3}} = \frac{5}{3} \times \log 2 = \frac{5}{3} \times .3010 = .5017.$$

To multiply a logarithm by a fraction, multiply first by the numerator, and divide the result by the denominator.

2. Given $\log 3 = .4771$; find $\log \sqrt[8]{3}$.

$$\log \sqrt[8]{3} = \frac{\log 3}{8} = \frac{.4771}{8} = .0596.$$

3. Given $\log 2 = .3010$, $\log 3 = .4771$, find $\log (2^{\frac{1}{3}} \times 3^{\frac{5}{4}})$.

By § 87, $\log (2^{\frac{1}{3}} \times 3^{\frac{5}{4}}) = \log 2^{\frac{1}{3}} + \log 3^{\frac{5}{4}}$

$$= \frac{1}{3} \log 2 + \frac{5}{4} \log 3 = .1003 + .5964 = .6967.$$

EXERCISE 14

Given $\log 2 = .3010$, $\log 3 = .4771$, $\log 7 = .8451$, find:

1. $\log 2^8$.
2. $\log 5^7$.
3. $\log 3^{\frac{4}{5}}$.
4. $\log 7^{\frac{2}{5}}$.
5. $\log 42^6$.
6. $\log 45^{\frac{1}{4}}$.
7. $\log 63^{\frac{1}{6}}$.
8. $\log 98^{\frac{7}{2}}$.
9. $\log 50^{\frac{3}{7}}$.
10. $\log \sqrt[5]{3}$.
11. $\log \sqrt[8]{5}$.
12. $\log \sqrt[13]{7}$.
13. $\log \sqrt[7]{8}$.
14. $\log \sqrt[4]{54}$.
15. $\log \sqrt[6]{225}$.
16. $\log \sqrt[9]{162}$.
17. $\log \sqrt[12]{\frac{7}{3}}$.
18. $\log (\frac{5}{2})^{\frac{1}{5}}$.
19. $\log (3^{\frac{3}{2}} \times 100^{\frac{1}{3}})$.
20. $\log (5\sqrt[11]{3})$.
21. $\log \dfrac{\sqrt[7]{7}}{\sqrt[3]{2}}$.
22. $\log \dfrac{2^{\frac{2}{5}}}{5^{\frac{1}{6}}}$.
23. $\log \dfrac{\sqrt[4]{35}}{7^{\frac{3}{8}}}$.
24. $\log \dfrac{3^{\frac{4}{7}}}{\sqrt[9]{75}}$.

92. In the common system, the mantissæ of the logarithms of numbers having the same sequence of figures are equal.

Suppose, for example, that $\log 3.053 = .4847$.

$$\text{Then, } \log 305.3 = \log(100 \times 3.053) = \log 100 + \log 3.053$$
$$= 2 + .4847 = 2.4847;$$
$$\log .03053 = \log(.01 \times 3.053) = \log .01 + \log 3.053$$
$$= 8 - 10 + .4847 = 8.4847 - 10; \text{ etc.}$$

It is evident from the above that, if a number be multiplied or divided by any integral power of 10, producing another number with the same sequence of figures, the mantissæ of their logarithms will be equal.

For this reason, only mantissæ are given, in a table of Common Logarithms; for to find the logarithm of any number, we have only to find the mantissæ corresponding to its sequence of figures, and then prefix the characteristic in accordance with the rules of §§ 82 and 83.

This property of logarithms only holds for the common system, and constitutes its superiority over other systems for numerical computation.

Whence, by § 83, log .00432 = 7.6353 – 10.

EXERCISE 15

Given log 2 = .3010, log 3 = .4771, log 7 = .8451, find:

1. log 2.7.
2. log 14.7.
3. log .56.
4. log .0162.
5. log 22.5.
6. log .00000686.
7. log .00125.
8. log 5670.
9. log .0000588.
10. log .000864.
11. log 337.5.
12. log 3.888.
13. log $(4.5)^8$.
14. log $\sqrt[6]{8.4}$.
15. log $(24.3)^{\frac{3}{5}}$.

USE OF THE TABLE

94. The table (pages 50 and 51) gives the mantissæ of the logarithms of all integers from 100 to 1000, calculated to four places of decimals.

95. *To find the logarithm of a number of three figures.*

Look in the column headed "No." for the first two significant figures of the given number.

Then the required mantissa will be found in the corresponding horizontal line, in the vertical column headed by the third figure of the number.

Finally, prefix the characteristic in accordance with the rules of §§ 82 and 83.

For example, log 168 = 2.2253;
log .344 = 9.5366 – 10; etc.

For a number consisting of one or two significant figures, the column headed 0 may be used.

Thus, let it be required to find log 83 and log 9.

By § 92, log 83 has the same mantissa as log 830, and log 9 the same mantissa as log 900.

Hence, log 83 = 1.9191, and log 9 = 0.9542.

96. *To find the logarithm of a number of more than three figures.*

1. Required the logarithm of 327.6.

We find from the table, $\log 327 = 2.5145$,
$\log 328 = 2.5159$.

That is, an increase of one unit in the number produces an increase of .0014 in the logarithm.

Then an increase of .6 of a unit in the number will increase the logarithm by $.6 \times .0014$, or .0008 to the nearest fourth decimal place.

Whence, $\log 327.6 = 2.5145 + .0008 = 2.5153$.

In finding the logarithm of a number, the difference between the next less and next greater mantissæ is called the *tabular difference;* thus, in Ex. 1, the tabular difference is .0014.

The subtraction may be performed mentally.

The following rule is derived from the above:

Find from the table the mantissa of the first three significant figures, and the tabular difference.

Multiply the latter by the remaining figures of the number, with a decimal point before them.

Add the result to the mantissa of the first three figures, and prefix the proper characteristic.

In finding the correction to the nearest units' figure, the decimal portion should be omitted, provided that if it is .5, or greater than .5, the units' figure is increased by 1; thus, 13.26 would be taken as 13, 30.5 as 31, and 22.803 as 23.

2. Find the logarithm of .021508.

$$\begin{array}{rl} \text{Mantissa } 215 = & .3324 \\ & \underline{2} \\ & .3326 \end{array} \qquad \begin{array}{rl} \text{Tab. diff.} = & 21 \\ & \underline{.08} \\ \text{Correction} = & 1.68 = 2\text{, nearly.} \end{array}$$

The result is $8.3326 - 10$.

EXERCISE 16

Find the logarithms of the following:

1. 64.	5. 1079.	9. .00005023.	13. 7.3165.
2. 3.7.	6. .6757.	10. .0002625.	14. .019608.
3. 982.	7. .09496.	11. 31.393.	15. 810.39.
4. .798.	8. 4.288.	12. 48387.	16. .0025446.

No.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
	0	1	2	3	4	5	6	7	8	9

No.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
No.	0	1	2	3	4	5	6	7	8	9

97. *To find the number corresponding to a logarithm.*

1. Required the number whose logarithm is 1.6571.

Find in the table the mantissa 6571.

In the corresponding line, in the column headed "No.," we find 45, the first two figures of the required number, and at the head of the column we find 4, the third figure.

Since the characteristic is 1, there must be two places to the left of the decimal point (§ 82).

Hence, the number corresponding to 1.6571 is 45.4.

2. Required the number whose logarithm is 2.3934.

We find in the table the mantissæ 3927 and 3945.

The numbers corresponding to the logarithms 2.3927 and 2.3945 are 247 and 248, respectively.

That is, an increase of .0018 in the mantissa produces an increase of one unit in the number corresponding.

Then, an increase of .0007 in the mantissa will increase the number by $\frac{7}{18}$ of a unit, or .4, nearly.

Hence, the number corresponding is 247 + .4, or 247.4.

The following rule is derived from the above:

Find from the table the next less mantissa, the three figures corresponding, and the tabular difference.

Subtract the next less from the given mantissa, and divide the remainder by the tabular difference.

Annex the quotient to the first three figures of the number, and point off the result.

The rules for pointing off are the reverse of those of §§ 82 and 83:

I. *If* − 10 *is not written after the mantissa, add* 1 *to the characteristic, giving the number of places to the left of the decimal point.*

II. *If* − 10 *is written after the mantissa, subtract the positive part of the characteristic from* 9, *giving the number of ciphers to be placed between the decimal point and first significant figure.*

3. Find the number whose logarithm is 8.5265 − 10.

$$\begin{array}{rl} & 5265 \\ \text{Next less mant.} = & \underline{5263}\text{; figures corresponding, 336.} \\ \text{Tab. diff.} & 13)2.00(.15 = .2\text{, nearly.} \\ & \underline{1\ 3} \\ & \ \ 70 \end{array}$$

By the above rule, there will be one cipher to be placed between the decimal point and first significant figure; the result is .03362.

The correction can usually be depended upon to only one decimal place; the division should be carried to two places to determine the last figure accurately.

EXERCISE 17

Find the numbers corresponding to the following logarithms:

1. 0.8189.
2. 7.6064 − 10.
3. 1.8767.
4. 2.6760.
5. 3.9826.
6. 8.7954 − 10.
7. 6.5993 − 10.
8. 9.9437 − 10.
9. 0.7781.
10. 5.4571 − 10.
11. 1.3019.
12. 4.2527 − 10.
13. 2.0159.
14. 3.7264 − 10.
15. 4.4929.

APPLICATIONS

98. The *approximate* value of a number in which the operations indicated involve only multiplication, division, involution, or evolution may be conveniently found by logarithms.

The utility of the process consists in the fact that addition takes the place of multiplication, subtraction of division, multiplication of involution, and division of evolution.

1. Find the value of $.0631 \times 7.208 \times .51272$.

By § 87, $\log(.0631 \times 7.208 \times .51272)$
$= \log .0631 + \log 7.208 + \log .51272.$

$$\begin{aligned} \log .0631 &= 8.8000 - 10 \\ \log 7.208 &= 0.8578 \\ \log .51272 &= \underline{9.7099 - 10} \end{aligned}$$

Adding, log of result $= 19.3677 - 20 = 9.3677 - 10$. (See Note 1.)

Number corresponding to $9.3677 - 10 = .2332$.

Note 1: If the sum is a negative logarithm, it should be written in such a form that the negative portion of the characteristic may be − 10.

Thus, 19.3677 − 20 is written 9.3677 − 10.

(In computations with four-place logarithms, the result cannot usually be depended upon to more than *four* significant figures.)

2. Find the value of $\frac{336.8}{7984}$.

$$
\begin{array}{lrl}
\text{By § 88,} & \log \frac{336.8}{7984} &= \log 336.8 - \log 7984. \\
& \log 336.8 &= 12.5273 - 10 \\
& \log\ 7984 &= \underline{\ \ 3.9022\qquad\ } \\
\text{Subtracting,} & \text{log of result} &= 8.6251 - 10 \qquad \text{(See Note 2.)} \\
& \text{Number corresponding} &= .04218.
\end{array}
$$

Note 2: To subtract a greater logarithm from a less, or a negative logarithm from a positive, increase the characteristic of the minuend by 10, writing -10 after the mantissa to compensate.

Thus, to subtract 3.9022 from 2.5273, write the minuend in the form $12.5273 - 10$; subtracting 3.9022 from this, the result is $8.6251 - 10$.

3. Find the value of $(.07396)^5$.

$$
\begin{array}{lrl}
\text{By § 89,} & \log (.07396)^5 &= 5 \times \log .07396. \\
& \log .07396 &= 8.8690 - 10 \\
& & \underline{\qquad\qquad\quad 5} \\
& & 44.3450 - 50 \\
& &= 4.3450 - 10 = \log .000002213.
\end{array}
$$

4. Find the value of $\sqrt[3]{.035063}$.

$$
\begin{array}{lrl}
\text{By § 90,} & \log \sqrt[3]{.035063} &= \tfrac{1}{3} \log .035063. \\
& \log .035063 &= 8.5449 - 10 \\
& & 3\underline{)28.5449 - 30} \qquad \text{(See Note 3.)} \\
& & \ \ 9.5150 - 10 = \log .3224.
\end{array}
$$

Note 3: To divide a negative logarithm, write it in such a form that the negative portion of the characteristic may be exactly divisible by the divisor, with -10 as the quotient.

Thus, to divide $8.5449 - 10$ by 3, we write the logarithm in the form $28.5449 - 30$; dividing this by 3, the quotient is $9.5150 - 10$.

EXERCISE 18

A *negative* number has no common logarithm (§ 78); if such numbers occur in computation, they may be treated as if they were positive, and the *sign* of the result determined irrespective of the logarithmic work.

Thus, in Ex. 3 of the following set, to find the value of $(-95.86) \times 3.3918$ we find the value of 95.86×3.3918, and put a $-$ sign before the result.

Find by logarithms the values of the following:

1. 4.253×7.104.
2. $6823.2 \times .1634$.
3. $(-95.86) \times 3.3918$.
4. $54.029 \times (-.0081487)$.
5. $.040764 \times .12896$.
6. $(-285.46) \times (-.00070682)$.
7. $\frac{5978}{9.762}$.
8. $\frac{21.658}{45057}$.
9. $\frac{.06405}{.002037}$.
10. $\frac{-38.19}{.10792}$.
11. $\frac{670.43}{-5382.3}$.
12. $\frac{.000007913}{.00082375}$.
13. $(88.08)^3$.
14. $(.09437)^4$.
15. $(3.625)^7$.

Arithmetical Complement

99. The **Arithmetical Complement** of the logarithm of a number, or, briefly, the **Cologarithm** of the number, is the logarithm of the reciprocal of that number.

Thus,
$$\text{colog } 409 = \log \tfrac{1}{409} = \log 1 - \log 409.$$
$$\log 1 = 10. \quad -10 \quad \text{(See Ex. 2, § 98.)}$$
$$\log 409 = 2.6117$$
$$\therefore \text{colog } 409 = 7.3883 - 10.$$

Again,
$$\text{colog } .067 = \log \frac{1}{.067} = \log 1 - \log .067$$
$$\log 1 = 10. \quad -10$$
$$\log .067 = 8.8261 - 10$$
$$\therefore \text{colog } .067 = 1.1739.$$

It follows from the above that *the cologarithm of a number may be found by subtracting its logarithm from* $10 - 10$.

The cologarithm may be found by subtracting the last *significant* figure of the logarithm from 10 and each of the others from 9, -10 being written after the result in the case of a positive logarithm.

Ex. Find the value of $\dfrac{.51384}{8.708 \times .0946}$.

$$\log \frac{.51384}{8.708 \times .0946} = \log\left(.51384 \times \frac{1}{8.708} \times \frac{1}{.0946}\right)$$
$$= \log .51384 + \log \frac{1}{8.708} + \log \frac{1}{.0946}$$
$$= \log .51384 + \text{colog } 8.708 + \text{colog } .0946.$$

$$\begin{aligned} \log .51384 &= 9.7109 - 10 \\ \text{colog } 8.708 &= 9.0601 - 10 \\ \text{colog } .0946 &= \underline{1.0241 \qquad\qquad\qquad\quad} \\ & \quad 9.7951 - 10 = \log .6239. \end{aligned}$$

It is evident from the above example that, to find the logarithm of a fraction whose terms are the products of factors, we **add together the logarithms of the factors of the numerator, and the cologarithms of the factors of the denominator.**

The value of the above fraction may be found without using cologarithms, by the following formula:

$$\log \frac{.51384}{8.709 \times .0946} = \log .51384 - \log(8.709 \times .0946)$$
$$= \log .51384 - (\log 8.709 + \log .0946).$$

The advantage in the use of cologarithms is that the written work of computation is exhibited in a more compact form.

MISCELLANEOUS EXAMPLES

100. 1. Find the value of $\dfrac{2\sqrt[3]{5}}{3^{\frac{5}{6}}}$.

$$\log \frac{2\sqrt[3]{5}}{3^{\frac{5}{6}}} = \log 2 + \log \sqrt[3]{5} + \text{colog } 3^{\frac{5}{6}} \qquad (\S\ 99)$$
$$= \log 2 + \tfrac{1}{3} \log 5 + \tfrac{5}{6} \text{ colog } 3.$$

$$\begin{aligned} && \log 2 &= .3010 \\ \log 5 = .6990; && \div 3 &= .2330 \\ \text{colog } 3 = 9.5229 - 10; && \times \tfrac{5}{6} &= \underline{9.6024 - 10} \\ && & .1364 \qquad = \log 1.369. \end{aligned}$$

2. Find the value of $\sqrt[3]{\dfrac{-.03296}{7.962}}$.

$$\log\sqrt[3]{\frac{.03296}{7.962}}=\frac{1}{3}\log\frac{.03296}{7.962}=\frac{1}{3}(\log .03296-\log 7.962).$$

$$\begin{aligned}\log .03296 &= 8.5180-10\\ \log\ \ 7.962 &= 0.9010\\ 3)\underline{27.6170-30}&\\ 9.2057-10 &= \log .1606.\end{aligned}$$

The result is $-.1606$.

EXERCISE 19

Find by logarithms the values of the following:

1. $\dfrac{2078.5\times .05834}{.3583\times 346}$.

2. $\dfrac{(-6.08)\times .1304}{4.046\times .0031095}$.

3. $\dfrac{(-.076917)\times 26.3}{.5478\times(-3120.7)}$.

4. $\dfrac{.8102\times(-6.225)}{(-0721)\times(-17.976)}$.

5. $6^{\frac{6}{5}}\times 5^{\frac{5}{6}}$.

6. $\dfrac{7^{\frac{4}{9}}}{9^{\frac{3}{8}}}$.

7. $\sqrt[11]{\frac{68}{35}}$.

8. $\dfrac{\sqrt[7]{8}}{(.1)^{\frac{2}{3}}}$.

9. $\dfrac{(100)^{\frac{2}{7}}}{\sqrt[3]{-.004}}$.

10. $\left(-\frac{5510}{7048}\right)^{\frac{4}{5}}$.

11. $\sqrt{\dfrac{38.7}{501.9}}$.

12. $\dfrac{\sqrt[5]{-.01}}{4^{\frac{9}{4}}}$.

13. $\dfrac{-(.03)^{\frac{5}{2}}}{\sqrt[9]{-1000}}$.

14. $\sqrt[4]{\frac{7}{9}}\div\sqrt[8]{\frac{3}{4}}$.

15. $\sqrt{6}\times\sqrt[6]{10}\times\sqrt[10]{2}$.

16. $\left(-\dfrac{24.18}{8.7\times .0603}\right)^{\frac{3}{5}}$.

17. $\dfrac{\sqrt[5]{.008546}}{\sqrt[6]{.0003867}}$.

18. $\dfrac{(-.14582)^{\frac{5}{3}}}{-(.72346)^{\frac{7}{4}}}$.

IV. FACTORS

101. An **irrational number** is a numerical expression involving surds; as $\sqrt[3]{3}$, or $2+\sqrt{5}$ (§ 70).

102. A rational and integral expression is resolved into its prime factors when further factoring would produce irrational factors.

103. In the First Course we considered the following eight types of factorable numbers:

TYPE FORMS

I. $a^2 - b^2 = (a+b)(a-b)$.

II. $a^2 + 2ab + b^2 = (a+b)(a+b)$,
$a^2 - 2ab + b^2 = (a-b)(a-b)$.

III. $x^2 + ax + b$.

IV. $ax^2 + bx + c$.

V. $x^4 + ax^2y^2 + y^4$.

VI. $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$,
$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$.

VII. $a^n - b^n$,
$a^n + b^n$.

VIII. $ax + ay + az = a(x + y + z)$.

Of these types, IV is more readily factored by means of VIII as follows:

Ex. Factor $6x^2 - 7x - 20$.

Multiply -20 by 6 (the coefficient of x^2). Factor -120 so that the sum of the factors is -7 (the coefficient of x). These factors are -15, 8. Then write

$$6x^2 - 7x - 20 = 6x^2 - 15x + 8x - 20.$$

Group by Type VIII, $\quad = 3x(2x-5) + 4(2x-5)$,

whence, $\quad 6x^2 - 7x - 20 = (2x-5)(3x+4)$.

Type VI may be placed under Type VII.

EXERCISE 20

Factor:

1. $3x^2 - x - 10$.
2. $4a^2 + 12a + 9$.
3. $x^3 - y^3$.
4. $a^3 + a^2 - 2a - 2$.
5. $x^4 + 4$.
6. $x^3 + 8$.
7. $a^2 + 9b^2 - 4c^2 + 6ab$.
8. $x^2 + 2xy + y^2 + 8(x+y) + 16$.

9. $x^{10} - 2x^5 + 1.$

10. $6a^2 - 17a + 12.$

11. $9x^4 - 13x^2 + 4.$

12. $x^6 + 7x^3 - 8.$

13. $(a-b)^2 - 2(a-b) - 35.$

14. $m^2 + (a-b)m - ab.$

15. $m^{\frac{3}{2}} - 1.$

16. $(2a - 3b)^2 - (a-b)^2.$

17. $p^{-2} - 7p^{-1} + 12.$

18. $a^3 + b + a + b^3.$

19. $x^3 + 3x^2 + 3x + 1.$

20. $9m^2 - 36mn.$

21. $x^9 - x^3 + x^6 - 1$

22. $6a^2b - 4a^2 + 15ab - 10a.$

23. $a^{-\frac{5}{2}} - 32.$

24. $9x^{-4} + 12x^{-2} + 4.$

25. $9a^2 - 30ab + 25b^2 - 4c^2.$

26. $9a^2 - 25c^2 + b^2 + 6ab.$

27. $36x^4 - 61x^2 + 25.$

28. $(3a-b)^2 - 6x(3a-b) + 27a - 9b.$

29. $2a^{-6} + 250.$

30. $(7x+2) + 3\sqrt{7x+2} + 2.$

31. $m(2x-3) - 4m^2x^2 + 9m^2.$

32. $x^6 + y^6.$

33. $16x^{\frac{4}{3}} + 14x^{\frac{2}{3}} - 15.$

34. $25(m+3)^2 + 10(m+3) + 1.$

35. $8(2a-5b)^{-1} - 12(2a-5b)^{-\frac{1}{2}} + 4.$

36. $143k^2 - 103k + 14.$

37. $\sqrt{x^2 + 4x - 6} + 2x^2 - 1 + 4(2x - 3).$

38. $g^4 + g^2t^2 + t^4.$

39. $x^3 + a^2x - a^3 - ax^2.$

40. $c^{-\frac{1}{3}}d - 18 - 9d + 2c^{-\frac{1}{3}}.$

41. $r^4 - 20r^2 + 99.$

42. $x^{12} + y^{12}.$

43. $x^4 - 13x^2 + 4.$

44. $9e^3f - 16ef^3.$

45. $304v^2 + 25v - 6.$

46. $(x+1)^{\frac{2}{3}} + 2(x+1)^{\frac{1}{3}} + 1.$

47. $5(x^2+y^2)^3 + 6(x^2+y^2)^2 + (x^2+y^2).$

48. $a^6 - 64.$

49. $6x^{-4} - 41x^{-2}y^{\frac{1}{2}} - 7y.$

50. $25a^4 + a^2 + 1.$

51. $4 + a^3 - a^2 - 4a.$

52. $m^4 - 1 + m - m^3.$

53. $3(x^3+1) + 5(x^2-1) + (x+1)^2.$

54. $6x^{-2} + 13x^{-1} + 6.$

55. $a^{\frac{5}{3}} + b^{\frac{5}{3}}.$

56. $x^{10} - y^5z^5.$

57. $p^2 - q.$

58. $4a^{2x+4} - 4a^{x+2} + 1.$

59. $a^2x - 9x + 2a^2 - 18.$

60. $4p^2q^2 + 20pq^2 - 16p^2q - 80pq$.

61. $(x^2 - 2x + 1) - (x + 1)^2$.

62. $52m - 10m^2 - 10$.

63. $x^{3m} - y^{3n}$.

64. $a^8 - 256$.

65. $a^5 - 27a^2 + 243 - 9a^3$.

66. $2^{2m} + 4x \cdot 2^m - 21x^2$.

67. $a + 2\sqrt{ab} + b$.

68. $(2a - 3b)^2 - (3a - 2b)^2$.

69. $a^2 + 2ab + c^2 - 2bc - 2ac + b^2$.

70. $27m^3 - 54m^2 + 36m - 8$.

71. $x^5 + x^4 + x^3 + x^2 + x + 1$.

72. $9a^2 - 6a - 4b^2 - 4b$.

73. $1 + 2ab - a^4 - a^2b^2 - b^4$.

74. $am^2 - m^3 + 2amn + an^2 + 2m^2n - mn^2$.

75. $y^2 + y^2z - 2y - z + 1$.

FACTOR THEOREM

104. The Remainder Theorem.

Let it be required to divide $px^2 + qx + r$ by $x - a$.

$$\begin{array}{l|l}
px^2 + qx + r & x - a \\ \cline{2-2}
\underline{px^2 - apx} & px + (ap + q) \\
(ap + q)x & \\
\underline{(ap + q)x - pa^2 - qa} & \\
\qquad\qquad pa^2 + qa + r, \text{ Remainder.} &
\end{array}$$

We observe that the final remainder,

$$pa^2 + qa + r,$$

is the same as the dividend with a substituted in place of x; this exemplifies the following law:

If any polynomial, involving x, be divided by $x - a$, the remainder of the division equals the result obtained by substituting a for x in the given polynomial.

This is called *The Remainder Theorem.*

To prove the theorem, let

$$px^n + qx^{n-1} + \cdots + rx + s$$

be any polynomial involving x.

Let the division of the polynomial by $x - a$ be carried on until a remainder is obtained which does not contain x.

Let Q denote the quotient, and R the remainder.

Since the dividend equals the product of the quotient and divisor, plus the remainder, we have

$$Q(x-a)+R=px^n+qx^{n-1}+\cdots+rx+s.$$

Putting x equal to a, into the above equation, we have,

$$R=pa^n+qa^{n-1}+\cdots+ra+s.$$

105. The Factor Theorem.

If any polynomial, involving x, becomes zero when x is put equal to a, the polynomial has $x-a$ as a factor.

For, by § 104, if the polynomial is divided by $x-a$, the remainder is zero.

106. Examples.

1. Find whether $x-2$ is a factor of x^3-5x^2+8.

Substituting 2 for x, the expression x^3-5x^2+8 becomes

$$2^3-5\cdot 2^2+8, \text{ or } -4.$$

Then, by § 104, if x^3-5x^2+8 be divided by $x-2$, the remainder is -4; and $x-2$ is not a factor.

2. Find whether $m+n$ is a factor of

$$m^4-4m^3n+3m^2n^2+5mn^3-2n^4. \qquad (1)$$

Putting $m=-n$, the expression becomes

$$n^4+4n^4+2n^4-5n^4-2n^4, \text{ or } 0.$$

Then, by § 104, if the expression (1) be divided by $m+n$, the remainder is 0; and $m+n$ is a factor.

3. Prove that a is a factor of

$$(a+b+c)(ab+bc+ca)-(a+b)(b+c)(c+a).$$

Putting $a=0$, the expression becomes

$$(b+c)bc-b(b+c)c, \text{ or } 0.$$

Then, by § 104, $a-0$, or a, is a factor of the expression.

4. Factor $x^3 - 3x^2 - 14x - 8$.

The positive and negative integral factors of 8 are 1, 2, 4, 8, -1, -2, -4, and -8.

It is best to try the numbers in their order of absolute magnitude.

If $x = 1$, the expression becomes $1 - 3 - 14 - 8$.

If $x = -1$, the expression becomes $-1 - 3 + 14 - 8$.

If $x = 2$, the expression becomes $8 - 12 - 28 - 8$.

If $x = -2$, the expression becomes $-8 - 12 + 28 - 8$, or 0.

This shows that $x + 2$ is a factor.

Dividing the expression by $x + 2$, the quotient is $x^2 - 5x - 4$.

Then, $$x^3 - 3x^2 - 14x - 8 = (x + 2)(x^2 - 5x - 4).$$

EXERCISE 21

Factor the following:

1. $a^3 + 8$.
2. $m^5 + n^5$.
3. $x^6 - 729$.
4. $x^3 + 5x^2 - 8x + 2$.
5. $m^3 - 11m - 10$.
6. $a^4 - a^3 + 3a - 14$.
7. $c^3 - 2c^2 - 9$.
8. $x^4 - 625$.
9. $a^n - b^n$.
10. $2x^3 + 5x^2 - x - 6$.
11. $x^4 - x^3 + 2x^2 - 4$.
12. $5a^3 - 18a - 4$.
13. $x^3 + x^2 + 7x + 18$.
14. $m^3 - 5m^2 - 36$.
15. $k^4 - 5k^2 + 3k - 2$.

Find without actual division:

16. Whether $p - 1$ is a factor of $p^3 + 3p^2 - 4$.
17. Whether $x + 2$ is a factor of $x^4 + 3x^3 - 4x$.
18. Whether $x + 1$ is a factor of $2x^3 + 5x^2 - 3x + 4$.
19. Whether $m - 3$ is a factor of $m^3 - 4m - 15$.
20. Whether $a - 5$ is a factor of $a^3 - 3a^2 - 5a - 25$.
21. Whether $c - 2$ is a factor of $3c^3 - 9c^2 + 5c + 2$.
22. Whether a is a factor of $a(b - c) + b(c - a) + c(a - b)$.
23. Whether c is a factor of $a(b - c) + b(c - a) + c(a - b)$.
24. Whether $x + y$ is a factor of $x(2x + 3y) - y(3x + 2y)$.
25. Whether b is a factor of $a^2(b - c)^2 + b^2(c - a)^2 + c^2(a - b)^2$.

HORNER'S SYNTHETIC DIVISION

107. The method of synthetic division, or as it is sometimes known, the method of detached coefficients, greatly abridges the work of division, especially where binomial divisors are concerned.

108. Divide $x^3 - 11x^2 + 36x - 36$ by $x - 3$.

Writing dividend and divisor with coefficients only,

$$\begin{array}{rrrr|l}
1 - & 11 + & 36 - & 36 & 1 - 3 \\ \cline{5-5}
\underline{1 -} & \underline{3} & & & 1 - 8 + 12 \quad \textit{Quotient.} \\
- & 8 & & & \\
- & 8 + & 24 & & \\ \cline{1-4}
 & & + 12 - & 36 & \\
 & & \underline{+ 12 -} & \underline{36} & \\
\end{array}$$

Since the first term of each partial product is merely a repetition of the term immediately above, it may be omitted.

We may also change the sign of the second term of the divisor if the partial product is added instead of subtracted.

We then have

$$\begin{array}{rrrr|l}
1 - & 11 + & 36 - & 36 & 1 + 3 \\ \cline{5-5}
\underline{1 +} & \underline{3} & & & 1 - 8 + 12 \\
 & - 8 & & & \\
 & & \underline{- 24} & & \\
 & & + 12 & & \\
 & & & \underline{+ 36} & \\
\end{array}$$

Raise the numbers -24, 36 now in the oblique column and the work stands :

$$\begin{array}{rrrr|l}
1 - & 11 + & 36 - & 36 & + 3 \\ \cline{5-5}
+ & 3 - & 24 + & 36 & \\ \cline{1-3}
- & & + 12 & & \\
\end{array}$$

The quotient is $x^2 - 8x + 12$.

If the last remainder is zero, x minus the divisor is a factor of the expression.

EXERCISE 22

Divide the following by synthetic division:

1. $2x^3 - 7x^2 + x + 10$ by $x - 2$.

2. $3a^4 - a^3 - 5a^2 + 6a + 7$ by $a + 1$.

3. $a^4 - 11a^3 + 29a^2 - 9a + 14$ by $a - 7$.

4. $4m^3 - 17m^2n + 13mn^2 + 6n^3$ by $m - 3n$.

5. $3x^5 + 11x^4 - 43x^3 - 4x^2 + 11x - 6$ by $x + 6$.

6. $8v^4 - 35v^3 + 7v^2 + 22v - 8$ by $v - 4$.

109. Divide $x^3 - 11x^2 + 36x - 36$ by $x - 5$, and by $x - 7$.

$$\begin{array}{rrrr|l} 1 & -11 & +36 & -36 & \underline{5} \\ & +5 & -30 & +30 & \\ \hline & -6 & +6 & -6 & \text{Remainder} \\ \multicolumn{4}{l}{\text{(Quotient)}} & \end{array} \qquad \begin{array}{rrrr|l} 1 & -11 & +36 & -36 & \underline{7} \\ & +7 & -28 & +56 & \\ \hline & -4 & +8 & +20 & \text{Remainder} \\ \multicolumn{4}{l}{\text{(Quotient)}} & \end{array}$$

A factor lies between $x - 5$ and $x - 7$. It is found to be $x - 6$.

Then if in dividing by a binomial a remainder occurs, and if the remainders arising from successive division by two binomials are of opposite sign, a factor $x - a$ lies between these two binomials.

EXERCISE 23

1. Locate the root between 2 and 4 of $x^3 - 17x + 24 = 0$.

Locate roots of the following:

2. $a^3 + 10a^2 + 17a - 28 = 0$.

3. $a^4 + 3a^3 - 10a^2 + 3a + 15 = 0$.

4. $x^5 - 8x^4 - 7x^3 + 56x^2 - 5x + 40 = 0$.

5. $3x^3 - 26x^2 + 60x - 72 = 0$.

6. $m^4 - 2m^3 - 19m^2 + 12m + 40 = 0$.

SOLUTIONS

110. If the product of $abc \cdots$ to n factors $= 0$, at least one of the factors must be zero.

Ex. 1. Let $(x-2)(x-3)(x+4)=0.$

Then $x-2$, $x-3$, or $x+4$ must equal zero.

The equation is satisfied by the root obtained by putting any one of the factors equal to 0. Hence, $x=2, 3$, or -4 are the solutions of the equation.

Ex. 2. Solve $$5^{2x}-5^x-12=0. \quad (1)$$

$$(5^x-4)(5^x+3)=0. \quad (2)$$

Whence, $$5^x-4=0, \quad 5^x=4, \quad (3)$$

and $$5^x+3=0, \quad 5^x=-3. \quad (4)$$

To solve (3) and (4), take the logarithms of each member of the equations:

From (3) $$x \log 5 = \log 4 \text{ (§ 89)}, \quad (5)$$

and $$x=\frac{\log 4}{\log 5}=\frac{.6020}{.6990}=\frac{602}{699}. \quad (6)$$

From (4) $$x \log 5 = -\log 3.$$

$$x=-\tfrac{4771}{6990}.$$

Ex. 3. Solve the equation $.2^x=3.$

Taking the logarithms of both members, $x \log .2 = \log 3$.

Then $$x=\frac{\log 3}{\log .2}=\frac{.4771}{9.3010-10}=\frac{.4771}{-.699}=-.6285+.$$

An equation of the form $a^x=b$ may be solved by inspection if b can be expressed as an exact power of a.

Ex. 4. Solve the equation $16^x=128.$

We may write the equation $(2^4)^x=2^7$, or $2^{4x}=2^7$.

Then, by inspection, $4x=7$; and $x=\frac{7}{4}$.

(If the equation were $16^x=\dfrac{1}{128}$, we could write it $(2^4)^x=\dfrac{1}{2^7}=2^{-7}$; then $4x$ would equal -7, and $x=-\frac{7}{4}$.)

EXERCISE 24

Solve the following equations:

1. $13^x = 8.$
2. $.06^x = .9.$
3. $9.347^x = .0625.$
4. $.005038^x = 816.3.$
5. $3^{4x-1} = 4^{2x+3}.$
6. $7^{3x+2} = .8^x.$
7. $.2^{x+5} = .5^{x-4}.$
8. $16^x = 32.$
9. $32^x = \frac{1}{64}.$
10. $(\frac{1}{16})^x = 8.$
11. $(\frac{1}{9})^x = \frac{1}{27}.$
12. $.04^{2x} - 5(.04)^x - 24 = 0.$
13. $2^{3x} + 7.2^{2x} - 9.2^x - 63 = 0.$
14. $3^{3y} - 5.3^{2y} - 8.3^y + 12 = 0.$
15. $11^{4x} - 5 \cdot 11^{2x} + 4 = 0.$
16. $2^{3x+3} - 6 \cdot 2^{2x+2} + 11 \cdot 2^{x+1} - 6 = 0.$
17. $.5^{4x} - 2(.5)^{3x} - 16(.5)^{2x} + 2(.5)^x + 15 = 0.$
18. $2^{3x} - 10 \cdot 2^{2x} - 71 \cdot 2^x - 60 = 0.$
19. $x^3 - x^2 - 9x + 9 = 0.$
20. $x^2 + (5c + 2d)x + 10cd = 0.$

COMMON FACTORS AND MULTIPLES

111. A **Common Factor** of two or more expressions is a factor of each of them.

112. The **Highest Common Factor** (H. C. F.) of two or more expressions is their common factor of *highest degree* (§ 23).

113. A **Common Multiple** of two or more expressions is an expression which is exactly divisible by each of them.

114. The **Lowest Common Multiple** (L. C. M.) of two or more expressions is their common multiple of *lowest degree.*

Ex. 1. Find the H. C. F. of $a^2 + 2a - 3$ and $1 - a^3$.

$$a^2 + 2a - 3 = (a-1)(a+3).$$
$$1 - a^3 = (1-a)(1+a+a^2).$$

The factors of the first expression can be put in the form

$$-(1-a)(3+a).$$

Hence, the H. C. F. is $1 - a$.

Ex. 2. Required the L. C. M. of

$$x^2-5x+6,\ x^2-4x+4, \text{ and } x^3-9x.$$

$$x^2-5x+6=(x-3)(x-2).$$
$$x^2-4x+4=(x-2)^2.$$
$$x^3-9x=x(x+3)(x-3).$$

It is evident by inspection that the L. C. M. of these expressions is

$$x(x-2)^2(x+3)(x-3).$$

115. When the polynomials cannot be readily factored by inspection, the H. C. F. and L. C. M. may be found by the following method.

The rule in Arithmetic for the H. C. F. of two numbers is:

Divide the greater number by the less.

If there be a remainder, divide the divisor by it; and continue thus to make the remainder the divisor, and the preceding divisor the dividend, until there is no remainder.

The last divisor is the H. C. F. required.

Thus, let it be required to find the H. C. F. of 169 and 546.

```
169)546(3
    507
     39)169(4
        156
         13)39(3
            39
```

Then 13 is the H. C. F. required.

116. We will now prove that a rule similar to that of § 115 holds for the H. C. F. of two algebraic expressions.

Let A and B be two polynomials, arranged according to the descending powers of some common letter.

Let the exponent of this letter in the first term of A be equal to, or greater than, its exponent in the first term of B.

Suppose that B is contained in A p times, with a remainder C; that C is contained in B q times, with a remainder D; and that D is contained in C r times, with no remainder.

To prove that D is the H. C. F. of A and B.

The operation of division is shown as follows:

$$\begin{array}{l} B)A(p \\ \underline{pB} \\ \quad C)B(q \\ \quad \underline{qC} \\ \quad\quad D)C(r \\ \quad\quad \underline{rD} \\ \quad\quad\quad 0 \end{array}$$

We will first prove that D is a common factor of A and B.

Since the minuend is equal to the subtrahend plus the remainder (F. C., § 40),

$$A = pB + C, \tag{1}$$

$$B = qC + D, \tag{2}$$

and

$$C = rD.$$

Substituting the value of C in (2), we obtain

$$B = qrD + D = D(qr + 1). \tag{3}$$

Substituting the values of B and C in (1), we have,

$$A = pD(qr + 1) + rD = D(pqr + p + r). \tag{4}$$

From (3) and (4), D is a common factor of A and B.

We will next prove that every common factor of A and B is a factor of D.

Let F be any common factor of A and B; and let

$$A = mF, \text{ and } B = nF.$$

From the operation of division, we have

$$C = A - pB, \tag{5}$$

and

$$D = B - qC. \tag{6}$$

Substituting the values of A and B in (5), we have

$$C = mF - pnF.$$

Substituting the values of B and C in (6) we have

$$D = nF - q(mF - pnF) = F(n - qm + pqn).$$

Whence, F is a factor of D.

Then, since every common factor of A and B is a factor of D, and since D itself is a common factor of A and B, it follows that D is the *highest* common factor of A and B.

We then have the following rule for the H. C. F. of two polynomials, A and B, arranged according to the descending powers of some common letter, the exponent of that letter in the first term of A being equal to, or greater than, its exponent in the first term of B:

Divide A by B.

If there be a remainder, divide the divisor by it; and continue thus to make the remainder the divisor, and the preceding divisor the dividend, until there is no remainder.

The last divisor is the H. C. F. required.

It is important to keep the work throughout in descending powers of some common letter; and each division should be continued until the exponent of this letter in the first term of the remainder is less than its exponent in the first term of the divisor.

Note 1: If the terms of one of the expressions have a common factor which is not a common factor of the terms of the other, it may be removed; for it can evidently form no part of the highest common factor.

In like manner, we may divide any remainder by a factor which is not a factor of the preceding divisor.

117. 1. Find the H. C. F. of

$$6x^2-25x+14 \text{ and } 6x^3-7x^2-25x+18.$$

$$\begin{array}{r|l} 6x^2-25x+14) & 6x^3-7x^2-25x+18(x+3 \\ & \underline{6x^3-25x^2+14x} \\ & 18x^2-39x \\ & \underline{18x^2-75x+42} \\ & 36x-24 \end{array}$$

In accordance with Note 1, we divide this remainder by 12, giving $3x-2$.

$$\begin{array}{r|l} 3x-2) & 6x^2-25x+14(2x-7 \\ & \underline{6x^2-4x} \\ & -21x \\ & \underline{-21x+14} \end{array}$$

Then, $3x-2$ is the H. C. F. required.

Note 2: If the first term of the dividend, or of any remainder, is not divisible by the first term of the divisor, it may be made so by multiplying the dividend or remainder by any term which is not a factor of the divisor.

2. Find the H. C. F. of

$$3a^3 + a^2b - 2ab^2 \text{ and } 4a^3b + 2a^2b^2 - ab^3 + b^4.$$

We remove the factor a from the first expression and the factor b from the second (Note 1), and find the H. C. F. of

$$3a^2 + ab - 2b^2 \text{ and } 4a^3 + 2a^2b - ab^2 + b^3.$$

Since $4a^3$ is not divisible by $3a^2$, we multiply the second expression by 3 (Note 2).

$$\begin{array}{r|rrrr|l}
 & 4a^3 + & 2a^2b - & ab^2 + & b^3 & \\
 & & & & 3 & \\
\hline
3a^2 + ab - 2b^2) & 12a^3 + & 6a^2b - & 3ab^2 + & 3b^3 & (4a \\
 & 12a^3 + & 4a^2b - & 8ab^2 & & \\
\hline
 & & 2a^2b + & 5ab^2 + & 3b^3 &
\end{array}$$

Since $2a^2b$ is not divisible by $3a^2$, we multiply this remainder by 3 (Note 2).

$$\begin{array}{r|rrr|l}
 & 2a^2b + & 5ab^2 + & 3b^3 & \\
 & & & 3 & \\
\hline
3a^2 + ab - 2b^2) & 6a^2b + & 15ab^2 + & 9b^3 & (2b \\
 & 6a^2b + & 2ab^2 - & 4b^3 & \\
\hline
 & & 13ab^2 + & 13b^3 &
\end{array}$$

We divide this remainder by $13b^2$ (Note 1), giving $a + b$.

$$\begin{array}{r|rrr|l}
a + b) & 3a^2 + & ab - & 2b^2 & (3a - 2b \\
 & 3a^2 + & 3ab & & \\
\hline
 & & -2ab & & \\
 & & -2ab - & 2b^2 & \\
\hline
\end{array}$$

Then, $a + b$ is the H. C. F. required.

Note 3: If the first term of any remainder is negative, the sign of each term of the remainder may be changed.

Note 4: If the given expressions have a common factor which can be seen by inspection, remove it, and find the H. C. F. of the resulting expressions; the result, multiplied by the common factor, will be the H. C. F. of the given expressions.

3. Find the H. C. F. of

$$2x^4+3x^3-6x^2+2x \text{ and } 6x^4+5x^3-2x^2-x.$$

Removing the common factor x (Note 4), we find the H. C. F. of

$$2x^3+3x^2-6x+2 \text{ and } 6x^3+5x^2-2x-1.$$

$$\begin{array}{r|l}
2x^3+3x^2-6x+2) & 6x^3+5x^2-2x-1\,(3 \\
 & \underline{6x^3+9x^2-18x+6} \\
 & \quad -4x^2+16x-7
\end{array}$$

The first term of this remainder being negative, we change the sign of each of its terms (Note 3).

$$\begin{array}{rl}
 & 2x^3+3x^2-6x+2 \\
 & \underline{\qquad\qquad\qquad\qquad 2} \\
4x^2-16x+7) & 4x^3+6x^2-12x+4\,(x \\
 & \underline{4x^3-16x^2+7x} \\
 & \qquad 22x^2-19x+4 \\
 & \qquad \underline{\qquad\qquad\quad 2} \\
 & \qquad 44x^2-38x+8\,(11 \\
 & \qquad \underline{44x^2-176x+77} \\
 & \qquad\quad 69)\underline{138x-69} \\
 & \qquad\qquad\quad 2x-1 \\
2x-1) & 4x^2-16x+7\,(2x-7 \\
 & \underline{4x^2-2x} \\
 & \quad -14x \\
 & \quad \underline{-14x+7}
\end{array}$$

The last divisor is $2x-1$; multiplying this by x, the H. C. F. of the given expressions is $x(2x-1)$.

(In the above solution, we multiply $2x^3+3x^2-6x+2$ by 2 in order to make its first term divisible by $4x^2$; and we multiply the remainder $22x^2-19x+4$ by 2 to make its first term divisible by $4x^2$.)

118. We will now show how to find the L. C. M. of two expressions which cannot be readily factored by inspection.

Let A and B be any two expressions.

Let F be their H. C. F., and M their L. C. M.

Suppose that $A=aF$, and $B=bF$.

Then, $$A\times B=abF^2. \qquad (1)$$

Since F is the H. C. F. of A and B, a and b have no common factors; whence the L. C. M. of aF and bF is abF.

That is, $$M=abF.$$

Multiplying each of these equals by F, we have

$$F \times M = abF^2. \qquad (2)$$

From (1) and (2), $\quad A \times B = F \times M.$

That is, *the product of two expressions is equal to the product of their* H. C. F. *and* L. C. M.

Therefore, to find the L. C. M. of two expressions,

Divide their product by their highest common factor; or,

Divide one of the expressions by their highest common factor, and multiply the quotient by the other expression.

Ex. Find the L. C. M. of

$$6x^2 - 17x^2 + 12 \text{ and } 12x^2 - 4x - 21.$$

$$\begin{array}{rl}
6x^2 - 17x + 12) & 12x^2 - 4x - 21(2 \\
 & \underline{12x^2 - 34x + 24} \\
 & 15)\underline{30x - 45} \\
 & \quad 2x - 3)6x^2 - 17x + 12(3x - 4 \\
 & \qquad\qquad \underline{6x^2 - 9x} \\
 & \qquad\qquad\quad -8x \\
 & \qquad\qquad\quad \underline{-8x + 12}
\end{array}$$

Then, the H.C.F. of the expressions is $2x - 3$.
Dividing $6x^2 - 17x + 12$ by $2x - 3$, the quotient is $3x - 4$.
Then, the L. C. M. is $(3x - 4)(12x^2 - 4x - 21)$.

EXERCISE 25

Find the H. C. F. and L. C. M. of the following:

1. $2a^2 + a - 6, \quad 4a^2 - 8a + 3.$
2. $6x^2 - 17x + 10, \quad 9x^2 - 14x - 8.$
3. $x^2 - 6x - 27, \quad x^3 - 2x^2 - 8x + 21.$
4. $6x^2 - 31xy + 18y^2, \quad 9x^2 + 15xy - 14y^2.$
5. $8x^2 + 6x - 9, \quad 6x^3 + 7x^2 - 7x - 6.$
6. $4x^2 - 11x - 3, \quad 8x^4 + 6x^3 - 11x^2 - 23x - 5.$
7. $m^5 - 4m^3 + m^2 - 4, \quad m^4 - 2m^3 - m^2 + m + 2.$
8. $12p^2 - 19pq - 21q^2, \quad 12p^3 + 5p^2q - 11pq^2 - 6q^3.$
9. $c^4 + 7c^3 + 12c^2, \quad c^3 + 4c^2 - 9c - 36, \quad 3c^3 + 10c^2 - 15c - 28.$

10. $8x^3+27,\ 4x^3-8x^2-9x+18,\ 2x^3+x^2-11x-12.$

11. $81-x^4,\ x^4-4x^3+4x^2-4x+3.$

12. $x^4+4,\ ax^2+2ax+2a,\ x^3+3x^2+4x+2.$

13. $16c^4+8c^2+81,\ 4c^3+4c^2+c-9.$

14. $a^3+7a^2-9a-63,\ a^3+6a^2+11a+6.$

15. $(5a-3b)^2-(a+b)^2,\ 72a^2-48ab+8b^2.$

16. $a^3+6a^2x+12ax^2+8x^3,\ 4a^5+8a^4x-a^3x^2-2a^2x^3.$

V. FRACTIONS

119. A **Fraction** is an indicated quotient written usually in the form $\frac{a}{b}$, where a is the dividend, and is called the numerator, and b the divisor, and called the denominator.

120. If the same factor be introduced into, or removed from, both dividend and divisor, the quotient is not changed. Upon this principle depends the reduction of fractions to either higher or lower terms. The laws of sign for fractions are those of ordinary division. The sign before the fraction denotes whether the quotient is to be added or subtracted.

REDUCTION OF FRACTIONS

121. Change of sign,

$$+\frac{+a}{+b}=-\frac{-a}{+b}=-\frac{+a}{-b}=+\frac{-a}{-b}.$$

EXERCISE 26

Write each of the following in three other ways without changing its value:

1. $\frac{a}{2}$. 2. $\frac{n+3}{7}$. 3. $\frac{8}{2-x}$. 4. $\frac{2x-7}{x+2}$. 5. $\frac{6x-5}{(x-3)(y+4)}$.

6. $\frac{b^2-a^2}{2b^2-a^2}$. 7. $\frac{(4x-3y)(y-3x)}{(2y+x)(x-y)}$.

122. Reduction to Lowest Terms. This is accomplished by removing every factor common to both numerator and denominator. If numerator and denominator are not prime to each other, it is possible generally to factor them by inspection. When, however, the factors cannot be readily seen, the method of § 117, known as the Euclidean method, may be used.

EXERCISE 27

Reduce the following to lowest terms:

1. $\dfrac{27x^3+8}{9x^2+12x+4}$.

2. $\dfrac{a^5-a^4b-ab^4+b^5}{a^4-a^3b-a^2b^2+ab^3}$.

3. $\dfrac{12z^2+16zy-3y^2}{10z^2+zy-21y^2}$.

4. $\dfrac{2x^3+5x^2-2x+3}{6x^3-7x^2+5x-2}$.

5. $\dfrac{x^3-x^2-4x-6}{x^3+7x^2+12x+10}$.

6. $\dfrac{16x^2+16x-32}{14x^2+14x-28}$.

Simplify the following:

7. $\left(\dfrac{a^2-3a+2}{a^2+5a+4}\right)\left(1+\dfrac{4a}{a^2-2a+1}\right)$.*

8. $\dfrac{x^2-y^2}{x^2+xy}+\dfrac{y}{x}-\dfrac{2x^3y}{x^2+y^2}$.

9. $\dfrac{a^2-15a+56}{c^3-125}\times\dfrac{c^2-3c-10}{c^2-8c+16}\div\dfrac{ac+2a-14-7c}{2c-8-4a+ac}$.

10. $\dfrac{1}{x+2}+\dfrac{1}{x-2}+\dfrac{2x}{x^2+4}+\dfrac{4x^3}{x^4+16}$.

11. $\dfrac{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}{\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a^2-b^2}{a^2+b^2}}$.

* To simplify this and following examples of Exercise 27, perform the indicated operations, then reduce the resulting fraction to its lowest terms.

12. $\dfrac{bc}{(a-b)(a-c)}-\dfrac{ab}{(c-a)(b-c)}-\dfrac{ac}{(c-b)(b-a)}$.

13. $\dfrac{4}{a-1}+\dfrac{5a^2-7a-6}{6a^2-a-12}\cdot\dfrac{8a^2-16a+6}{2a^2-5a+2}$.

14. $\dfrac{4}{3x-\dfrac{x-y}{1-\dfrac{2x-3y}{3x-4y}}}$.

15. $\dfrac{m-2}{m+5}-\dfrac{4-m}{3-m}+\dfrac{1-m}{m-5}$.

16. $\left(3a+5-\dfrac{a+7}{a+2}\right)\div\left(\dfrac{5a}{a^2-10a-24}+1\right)$.

17. $\left[\dfrac{1}{3x+2y}+\dfrac{1}{3x-2y}\right]\div\left[\dfrac{1}{27x^3+8y^3}+\dfrac{1}{27x^3-8y^3}\right]$.

18. $\dfrac{4c^4-29c^2+25}{c^6-1}\div\left(\dfrac{4c^2-20c+25}{3c^2+3c+3}\times\dfrac{6c^2+11c-10}{9c^2-4}\right)$.

19. $\dfrac{x-5+\dfrac{9(x^2+5x+6)}{x^3+6x^2+11x+6}}{\dfrac{x^4-2x^3-8x+16}{x^3+3x^2+3x+1}}$.

20. $\dfrac{\dfrac{1}{x}+\dfrac{2}{y}}{1+\dfrac{4xy+y^2}{4x^2}}-\dfrac{\dfrac{4x}{y}-2+\dfrac{y}{x}}{8x^3+y^3}$.

21. $\dfrac{x+4}{x+2}-\dfrac{x-1}{x-3}+\dfrac{x+2}{x-5}-\dfrac{x^2-x-16}{x^2-8x+15}$.

22. $\left(1-\dfrac{x^2+2x-11}{x^2+5x-14}\right)\div\dfrac{9}{x^3+343}$.

23. $\dfrac{2}{x+1}-\dfrac{x^2+3x+2}{x^2-1}\cdot\dfrac{4}{x^2+5x+6}+\dfrac{x^2+1}{x}\div\dfrac{x^4-1}{12x}.$

24. $\dfrac{(2x^2-2xy-2x)(x^2-y^2)}{[(x-y)^2-1]\div\dfrac{x}{x+y}}.$

25. $\dfrac{a^5+b^5}{a^2+b^2}\times\dfrac{a^4-b^4}{a^2b+ab^2}\times\dfrac{3a^2}{a^4-a^3b+a^2b^2-ab^3+b^4}.$

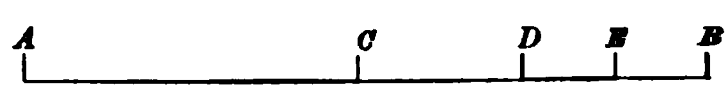
123. Under certain conditions a fraction may assume a form the value of which is not readily seen. Such forms usually occur in limiting values of fractions in which the unknown or unknowns are considered variable.

124. A *variable number*, or simply a *variable*, is a number which may assume, under the conditions imposed upon it, an indefinitely great number of different values.

A *constant* is a number which remains unchanged throughout the same discussion.

125. A *limit* of a variable is a constant number, the difference between which and the variable may be made less than any assigned number, however small.

Suppose, for example, that a point moves from A towards B under the condition that it shall move, during successive equal intervals of time, first from A to C, halfway between A and B; then to D, halfway between C and B; then to E, halfway between D and B; and so on indefinitely.



In this case, the distance between the moving point and B can be made less than any assigned number, however small.

Hence, the distance from A to the moving point is a variable which approaches the constant value AB as a limit.

Again, the distance from the moving point to B is a variable which approaches the limit 0.

126. Interpretation of $\frac{a}{0}$.

Consider the series of fractions $\dfrac{a}{3}, \dfrac{a}{.3}, \dfrac{a}{.03}, \dfrac{a}{.003}, \cdots.$

Here each denominator after the first is one-tenth of the preceding denominator.

It is evident that, by sufficiently continuing the series, the denominator may be made less than any assigned number, however small, and the value of the fraction greater than any assigned number, however great.

In other words,

If the numerator of a fraction remains constant, while the denominator approaches the limit 0, the value of the fraction increases without limit.

It is customary to express this principle as follows:

$$\frac{a}{0}=\infty.$$

The symbol ∞ is called *Infinity;* it simply stands for that which is greater than any number, however great, and has no fixed value.

127. Interpretation of $\frac{a}{\infty}$.

Consider the series of fractions $\frac{a}{3}, \frac{a}{30}, \frac{a}{300}, \frac{a}{3000}, \cdots$.

Here each denominator after the first is ten times the preceding denominator.

It is evident that, by sufficiently continuing the series, the denominator may be made greater than any assigned number, however great, and the value of the fraction less than any assigned number, however small.

In other words,

If the numerator of a fraction remains constant, while the denominator increases without limit, the value of the fraction approaches the limit 0.

It is customary to express this principle as follows:

$$\frac{a}{\infty}=0.$$

128. No *literal meaning* can be attached to such results as

$$\frac{a}{0}=\infty, \text{ or } \frac{a}{\infty}=0;$$

for there can be no such thing as division unless the divisor is a *finite number*.

If such forms occur in mathematical investigations, they must be interpreted as indicated in §§ 126 and 127. (Compare § 86.)

THE PROBLEM OF THE COURIERS

129. The following discussion will further illustrate the form $\frac{a}{0}$, besides furnishing an interpretation of the form $\frac{0}{0}$.

The Problem of the Couriers.

Two couriers, A and B, are travelling along the same road in the same direction, RR', at the rates of m and n miles an hour, respectively. If at any time, say 12 o'clock, A is at P, and B is a miles beyond him at Q, after how many hours, and how many miles beyond P, are they together?

Let A and B meet x hours after 12 o'clock, and y miles beyond P.

They will then meet $y-a$ miles beyond Q.

Since A travels mx miles, and B nx miles, in x hours, we have

$$\begin{cases} y = mx, \\ y-a = nx. \end{cases}$$

Solving these equations, we obtain

$$x=\frac{a}{m-n}, \text{ and } y=\frac{am}{m-n}.$$

We will now discuss these results under different hypotheses.

1. $m > n$.

In this case, the values of x and y are *positive*.

This means that the couriers meet at some time *after* 12, at some point to the *right* of P.

This agrees with the hypothesis made; for if m is greater than n, A is travelling faster than B; and he must overtake him at some point beyond their positions at 12 o'clock.

2. $m < n$.

In this case, the values of x and y are *negative*.

This means that the couriers met at some time *before* 12, at some point to the *left* of P.

This agrees with the hypothesis made; for if m is less than n, A is travelling more slowly than B; and they must have been together before 12 o'clock, and before they could have advanced as far as P.

3. $a = 0$, and $m > n$ or $m < n$.

In this case, $x = 0$ and $y = 0$.

This means that the travellers are together at 12 o'clock, at the point P.

This agrees with the hypothesis made; for if $a = 0$, and m and n are unequal, the couriers are together at 12 o'clock, and are travelling at unequal rates; and they could not have been together before 12, and will not be together afterwards.

4. $m = n$, and a not equal to 0.

In this case, the values of x and y take the forms $\frac{a}{0}$ and $\frac{am}{0}$, respectively.

If $m - n$ approaches the limit 0, the values of x and y increase without limit (§ 126); hence, if $m = n$, no fixed values can be assigned to x and y, and the problem is impossible.

In this case, *the result in the form* $\frac{a}{0}$ *indicates that the given problem is impossible.*

This agrees with the hypothesis made; for if $m = n$, and a is not zero, the couriers are a miles apart at 12 o'clock, and are travelling at the same rate; and they never could have been, and never will be together.

5. $m = n$, and $a = 0$.

In this case, the values of x and y take the form $\frac{0}{0}$.

If $a = 0$, and $m = n$, the couriers are together at 12 o'clock, and travelling at the same rate.

Hence, they always have been, and always will be, together.

In this case, the number of solutions is indefinitely great; for any value of x whatever, together with the corresponding value of y, will satisfy the given conditions.

In this case, *the result in the form* $\frac{0}{0}$ *indicates that the number of solutions is indefinitely great.*

Such form is called **Indeterminate.**

130. In § 129, we found that the form $\frac{0}{0}$ indicated an expression which could have *any value whatever;* but this is not always the case.

Consider, for example, the fraction $\dfrac{x^2 - a^2}{x^2 - ax}$.

If $x = a$, the fraction takes the form $\frac{0}{0}$.

Now, $$\frac{x^2 - a^2}{x^2 - ax} = \frac{(x + a)(x - a)}{x(x - a)} = \frac{x + a}{x};$$

which last expression is equal to the given fraction provided x does not equal a.

The fraction $\dfrac{x + a}{x}$ approaches the limit $\dfrac{a + a}{a}$, or 2, when x approaches the limit a.

This limit we call *the value of the given fraction when* $x = a$.

Then, the value of the given fraction when $x = a$ is 2.

In any similar case, we cancel the factor which equals 0 for the given value of x, and find the limit approached by the result when x approaches the given value as a limit.

EXERCISE 28

Find the values of the following:

1. $\dfrac{2\,ax - 4\,a^2}{x^2 - 4\,a^2}$ when $x = 2\,a$.

2. $\dfrac{2\,x^3 - 5\,x^2}{4\,x^2 + 3\,x}$ when $x = 0$.

3. $\dfrac{x^2 - 16}{x^2 + 2\,x - 8}$ when $x = -4$.

4. $\dfrac{4\,x^2 - 4\,x - 3}{6\,x^2 - 17\,x + 12}$ when $x = \frac{3}{2}$.

5. $\dfrac{x^3 + 6\,x^2 + 12\,x + 8}{x^4 - 8\,x^2 + 16}$ when $x = -2$.

6. $\dfrac{x^3 - 3\,x^2 + 3\,x - 2}{x^3 - 7\,x + 6}$ when $x = 2$.

131. Other Indeterminate Forms.

Expressions taking the forms $\frac{\infty}{\infty}$, $0 \times \infty$, or $\infty - \infty$, for certain values of the letters involved, are also indeterminate.

1. Find the value of $(x^3+8)\left(1+\dfrac{1}{x+2}\right)$ when $x=-2$.

This expression takes the form $0 \times \infty$, when $x=-2$ (§ 126).

Now, $(x^3+8)\left(1+\dfrac{1}{x+2}\right)=x^3+8+\dfrac{x^3+8}{x+2}$

$$=x^3+8+x^2-2x+4=x^3+x^2-2x+12.$$

The latter expression approaches the limit $-8+4+4+12$, or 12, when x approaches the limit -2.

This limit we call *the value of the expression when* $x=-2$; then, the value of the expression when $x=-2$, is 12.

In any similar case, we simplify as much as possible before finding the limit.

2. Find the value of $\dfrac{1}{1-x}-\dfrac{2x}{1-x^2}$ when $x=1$.

The expression takes the form $\infty-\infty$, when $x=1$ (§ 126).

Now, $$\frac{1}{1-x}-\frac{2x}{1-x^2}=\frac{1+x-2x}{1-x^2}=\frac{1-x}{1-x^2}=\frac{1}{1+x}.$$

The latter expression approaches the limit $\frac{1}{2}$ when x approaches the limit 1.

Then, the value of the expression when $x=1$, is $\frac{1}{2}$.

132. Another example in which the result is indeterminate is the following:

Ex. Find the limit approached by the fraction $\dfrac{1+2x}{2-5x}$ when x is indefinitely increased.

Both numerator and denominator increase indefinitely in absolute value when x is indefinitely increased.

Dividing each term of the fraction by x, $$\frac{1+2x}{2-5x}=\frac{\frac{1}{x}+2}{\frac{2}{x}-5}.$$

The latter expression approaches the limit $\dfrac{0+2}{0-5}$ (§ 127), or $-\dfrac{2}{5}$, when x is indefinitely increased.

In any similar case, we divide both numerator and denominator of the fraction by the highest power of x.

EXERCISE 29

Find the limits approached by the following when x is indefinitely increased:

1. $\dfrac{4+5x-3x^2}{7-x+4x^2}$. 2. $\dfrac{2x+1}{3x^2-2}$. 3. $\dfrac{x^3-2x-4}{x^2+5x+3}$.

Find the values of the following:

4. $\dfrac{1}{x-2}-\dfrac{12}{x^3-8}$ when $x=2$.

5. $(2x^2-5x-3)\left(2+\dfrac{1}{x-3}\right)$ when $x=3$.

RATIO AND PROPORTION

RATIO

133. The **Ratio** of one number a to another number b is the quotient of a divided by b.

Thus, the ratio of a to b is $\dfrac{a}{b}$; it is also expressed $a:b$.

The ratios here spoken of are but fractions under another name, and *have all the properties of fractions.*

In the ratio $a:b$, a is called the *first term*, or *antecedent*, and b the *second term*, or *consequent.*

If a and b are positive numbers, and $a>b$, $\dfrac{a}{b}$ is called a *ratio of greater inequality;* if $a<b$, it is called a *ratio of less inequality.*

134. A ratio of greater inequality is decreased, and one of less inequality is increased, by adding the same positive number to each of its terms.

Let a and b be positive numbers, a being $>b$, and x a positive number.

Since $a>b$, $\qquad ax>bx. \qquad$ (§ 59)

Adding ab to both members (§ 56),

$$ab+ax>ab+bx, \text{ or } a(b+x)>b(a+x).$$

Dividing both members by $b(b+x)$, we have

$$\frac{a}{b} > \frac{a+x}{b+x}. \qquad (\S\ 59)$$

In like manner, if $a < b$, $\quad \frac{a}{b} < \frac{a+x}{b+x}.$

PROPORTION

135. A **Proportion** is an equation whose members are equal ratios.

Thus, if $a : b$ and $c : d$ are equal ratios,

$$a : b = c : d, \text{ or } \frac{a}{b} = \frac{c}{d},$$

is a proportion. The latter form is preferable.

136. In the proportion $a : b = c : d$, a is called the *first term*, b the *second*, c the *third*, and d the *fourth*.

The first and third terms of a proportion are called the *antecedents*, and the second and fourth terms the *consequents*.

The first and fourth terms are called the *extremes*, and the second and third terms the *means*.

137. If the means of a proportion are equal, either mean is called the **Mean Proportional** between the first and last terms, and the last term is called the **Third Proportional** to the first and second terms.

Thus, in the proportion $a : b = b : c$, b is the mean proportional between a and c, and c is the third proportional to a and b.

The **Fourth Proportional** to three numbers is the fourth term of a proportion whose first three terms are the three numbers taken in their order.

Thus, in the proportion $a : b = c : d$, d is the fourth proportional to a, b, and c.

138. A **Continued Proportion** is a series of equal ratios, in which each consequent is the same as the next antecedent; as,

$$a : b = b : c = c : d = d : e.$$

PROPERTIES OF PROPORTIONS

139. In any proportion, the product of the extremes is equal to the product of the means.

Let the proportion be $\frac{a}{b}=\frac{c}{d}$.

Clearing of fractions, $ad = bc$.

140. From the equation $ad = bc$ (§ 139), we obtain

$$a=\frac{bc}{d},\ b=\frac{ad}{c},\ c=\frac{ad}{b},\text{ and } d=\frac{bc}{a}.$$

That is, in any proportion, either extreme equals the product of the means divided by the other extreme; and either mean equals the product of the extremes divided by the other mean.

141. (Converse of § 139.) **If the product of two numbers be equal to the product of two others, one pair may be made the extremes, and the other pair the means, of a proportion.**

Let $ad = bc$.

Dividing by bd, $\frac{ad}{bd}=\frac{bc}{bd}$, or $\frac{a}{b}=\frac{c}{d}$.

In like manner, we may prove that

$$\frac{a}{c}=\frac{b}{d},$$

$$\frac{c}{d}=\frac{a}{b},\text{ etc.}$$

142. In any proportion, the terms are in proportion by ***Alternation;*** that is, the means may be interchanged.

Let the proportion be $\frac{a}{b}=\frac{c}{d}$.

Then, by § 139, $ad = bc$.

Then, by § 141, $\frac{a}{c}=\frac{b}{d}$.

In like manner it may be proved that the extremes can be interchanged.

143. In any proportion, the terms are in proportion by *Inversion;* that is, the second term is to the first as the fourth term is to the third.

Let the proportion be $\frac{a}{b}=\frac{c}{d}$,

Then, by § 139, $ad = bc$.

Whence, by § 141, $\frac{b}{a}=\frac{d}{c}$.

It follows from § 143 that, in any proportion, the means can be written as the extremes, and the extremes as the means.

144. The mean proportional between two numbers is equal to the square root of their product.

Let the proportion be $\frac{a}{b}=\frac{b}{c}$.

Then, by § 139, $b^2 = ac$, or $b=\sqrt{ac}$.

145. In any proportion, the terms are in proportion by *Composition;* that is, the sum of the first two terms is to the first term as the sum of the last two terms is to the third term.

Let the proportion be $\frac{a}{b}=\frac{c}{d}$.

Then, $ad = bc$.

Adding each member of the equation to ac,

$$ac + ad = ac + bc, \text{ or } a(c+d) = c(a+b).$$

By § 141, $\frac{a+b}{a}=\frac{c+d}{c}$.

We may also prove $\frac{a+b}{b}=\frac{c+d}{d}$.

146. In like manner we may also prove that the terms of any proportion are in proportion by ***Division;*** that is, **the difference between the first two terms is to the first term as the difference between the last two terms is to the third term.**

The proof is left to the student.

147. In any proportion, the terms are in proportion by *Composition and Division*; that is, the sum of the first two terms is to their difference as the sum of the last two terms is to their difference.

The proof is left to the student. HINT. — Divide the result of § 145 by that of § 146.

148. In any proportion, if the first two terms be multiplied by any number, as also the last two, the resulting numbers will be in proportion.

Let the proportion be $\frac{a}{b}=\frac{c}{d}$; then, $\frac{ma}{mb}=\frac{nc}{nd}$.

(Either m or n may be unity; that is, the terms of either ratio may be multiplied without multiplying the terms of the other.)

149. In any proportion, if the first and third terms be multiplied by any number, as also the second and fourth terms, the resulting numbers will be in proportion.

Let the proportion be $\frac{a}{b}=\frac{c}{d}$; then, $\frac{ma}{nb}=\frac{mc}{nd}$.

(Either m or n may be unity.)

150. In any number of proportions, the products of the corresponding terms are in proportion.

Let the proportions be $\frac{a}{b}=\frac{c}{d}$, and $\frac{e}{f}=\frac{g}{h}$.

Multiplying, $\frac{a}{b}\times\frac{e}{f}=\frac{c}{d}\times\frac{g}{h}$, or $\frac{ae}{bf}=\frac{cg}{dh}$.

In like manner, the theorem may be proved for any number of proportions.

151. In any proportion, like powers or like roots of the terms are in proportion.

Let the proportion be $\frac{a}{b}=\frac{c}{d}$; then, $\frac{a^n}{b^n}=\frac{c^n}{d^n}$.

In like manner, $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\frac{\sqrt[n]{c}}{\sqrt[n]{d}}$.

152. In a series of equal ratios, any antecedent is to its consequent as the sum of all the antecedents is to the sum of all the consequents.

Let $a:b=c:d=e:f.$

Then, by § 139, $ad=bc,$

and $af=be.$

Also, $ab=ba.$

Adding, $a(b+d+f)=b(a+c+e).$

Whence, $a:b=a+c+e:b+d+f.$ (§ 141)

In like manner, the theorem may be proved for any number of equal ratios.

153. If three numbers are in continued proportion, the first is to the third as the square of the first is to the square of the second.

Let the proportion be $a:b=b:c$; or $\frac{a}{b}=\frac{b}{c}.$

Then, $\frac{a}{b}\times\frac{b}{c}=\frac{a}{b}\times\frac{a}{b}$, or $\frac{a}{c}=\frac{a^2}{b^2}.$

154. If four numbers are in continued proportion, the first is to the fourth as the cube of the first is to the cube of the second.

Let the proportion be $a:b=b:c=c:d$; or $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}.$

Then, $\frac{a}{b}\times\frac{b}{c}\times\frac{c}{d}=\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}$, or $\frac{a}{d}=\frac{a^3}{b^3}.$

Similarly, it may be shown that if n numbers are in continued proportion, the first antecedent is to the last consequent as the nth power of the first antecedent is to the nth power of its consequent.

155. Examples.

1. If $x:y=(x+z)^2:(y+z)^2$, prove z the mean proportional between x and y.

From the given proportion, by § 139,

$$y(x+z)^2=x(y+z)^2.$$

Or, $x^2y+2xyz+yz^2=xy^2+2xyz+xz^2.$

Transposing, $x^2y-xy^2=xz^2-yz^2.$

Dividing by $x-y$, $xy=z^2.$

Therefore, z is the mean proportional between x and y (§ 144).

The theorem of § 147 saves work in the solution of a certain class of fractional equations.

2. Solve the equation $\dfrac{2x+3}{2x-3}=\dfrac{2b-a}{2b+a}$.

Regarding this as a proportion, we have by composition and division,

$$\frac{4x}{6}=\frac{4b}{-2a}, \text{ or } \frac{2x}{3}=-\frac{2b}{a}; \text{ whence, } x=-\frac{3b}{a}.$$

3. Prove that if $\dfrac{a}{b}=\dfrac{c}{d}$, then

$$a^2-b^2 : a^2-3ab = c^2-d^2 : c^2-3cd.$$

Let $\dfrac{a}{b}=\dfrac{c}{d}=x$, whence, $a=bx$; then,

$$\frac{a^2-b^2}{a^2-3ab}=\frac{b^2x^2-b^2}{b^2x^2-3b^2x}=\frac{x^2-1}{x^2-3x}=\frac{\dfrac{c^2}{d^2}-1}{\dfrac{c^2}{d^2}-\dfrac{3c}{d}}=\frac{c^2-d^2}{c^2-3cd}.$$

Then, $$a^2-b^2 : a^2-3ab = c^2-d^2 : c^2-3cd.$$

EXERCISE 30

1. Find the mean proportional between .0289 and 1.69.
2. Find the mean proportional between $1\frac{75}{121}$ and $12\frac{24}{25}$.
3. Find the third proportional to $\frac{15}{16}$ and $1\frac{1}{8}$.
4. Find the fourth proportional to $9\frac{9}{80}$, $16\frac{1}{5}$, and $\frac{9}{16}$.
5. Find the fourth proportional to m, n, and r.
6. Write in the form of a proportion: $x^2-2x-15=a^2$.

Solve, using composition and division:

7. $\dfrac{4x+5}{4x-5}=\dfrac{x+5}{x-3}$.

8. $\dfrac{x+a}{x-a}=\dfrac{b+c}{b-c}$.

9. $\dfrac{a^{\frac{1}{2}}-x}{a^{\frac{1}{2}}+x}=\dfrac{1}{3}$.

10. $\dfrac{2x+7}{2x-3}=\dfrac{5x+1}{5x-9}$.

11. $\dfrac{(m+1)^{\frac{1}{2}}+(m-1)^{\frac{1}{2}}}{(m+1)^{\frac{1}{2}}-(m-1)^{\frac{1}{2}}}=3$.

12. If $\dfrac{a}{b}=\dfrac{b}{c}$, show that $a:c=b^2:c^2$.

13. If $a:b=c:d$, show that $\dfrac{4a^2c-6abc+9b^2c}{2a^2-3ab}=\dfrac{8c^3+27d^3}{4c^2-9d^2}$.

14. Find two numbers in the ratio of $2:3$, such that the sum of their squares shall be 208.

15. Find two numbers in the ratio of $3:1$, such that the difference of their squares is 200.

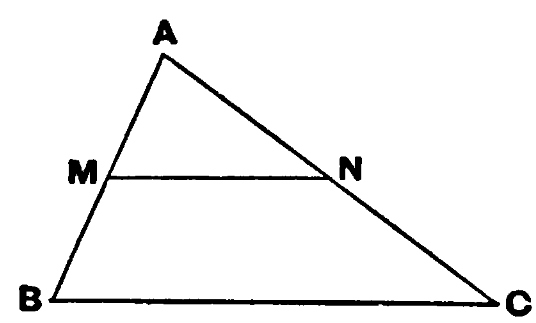
16. Two numbers are in the ratio of $5:7$. If 6 be added to each, they will be in the ratio of $7:9$. Find the numbers.

17. Two numbers are in the ratio of $2:5$. If 4 be added to each number, the resulting ratio will be twice the ratio had 4 been subtracted from each number. Find the numbers.

18. The difference between two numbers is 6, and the difference between their squares is 60. What is the ratio of their sum to their difference?

19. In similar figures in geometry, homologous sides are proportional. If a pole 30 feet high casts a shadow 42 feet long, how high must a pole be to cast a shadow 35 feet long?

20. A ladder 40 feet long leans against the side of a building, with its foot 12 feet from the building. A second ladder, $40\frac{1}{2}$ feet long, makes the same angle with the building as the first ladder. How far is the foot of the second ladder from the building?



21. In the triangle ABC, MN is drawn parallel to BC and divides the other two sides proportionally. If $AM=12$, $\dfrac{AM}{AN}=\dfrac{2}{3}$, and $BC=48$, how long is AC? (M is the middle point of AB.) What is the ratio of AN to MN?

22. The areas of any two similar figures are to each other as the squares of their homologous parts. If a regular hexagon has a side equal to 6 and an area of $54\sqrt{3}$, what is the area of a regular hexagon whose side is 2?

23. The area of a circle is $6\frac{1}{4}$ times that of another circle. If the radius of the first circle is 5, what is the radius of the second circle?

24. If the altitude of a triangle is twice that of a similar triangle, how do their areas compare?

25. The volume of a rectangular solid is equal to the product of its three dimensions, x, y, and z. If $xyz = v$ and $x : y : z = a : b : c$, find x, y, and z in terms of a, b, c, and v.

26. Find three numbers in continued proportion whose sum is 63, the second being 4 times the first.

27. Given the proportion $\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$, where $d = 81$ and $\frac{a}{b} = \frac{1}{3}$. Find a, b, and c.

28. If $2a - 3b : 4a - 5b = 2b - 3c : 4b - 5c$, prove b is the mean proportional to a and c.

29. If $3a + 5b : 4a - 7b = 3c + 5d : 4c - 7d$, prove $\frac{a}{b} = \frac{c}{d}$.

30. Find two numbers in the ratio of a to b, such that if each be increased by $\frac{c}{d}$ they will be in the ratio of e to f.

31. $\frac{4x+7}{5} - \frac{8x+4}{15} - \frac{12x+1}{45} = \frac{5x-1}{9(5x+2)}$. Solve for x.

32. $\frac{a+b}{x} + \frac{a-2b}{x+a} = \frac{(2a-b)x+3ab}{x^2-a^2}$. Solve for x.

33. A man borrows a certain sum, paying interest at the rate of 5%. After repaying \$180, his interest rate on the balance is reduced to $4\frac{1}{4}$%, and his annual interest is now less by \$10.80. Find the sum borrowed.

34. The digits of a certain number are three consecutive numbers, of which the middle digit is the greatest, and the first digit the least. If the number be divided by the sum of its digits, the quotient is $\frac{239}{7}$. Find the number.

35. A certain number of apples were divided between three boys. The first received one-half the entire number, with one apple additional, the second received one-third the remainder, with one apple additional, and the third received the remainder, 7. How many apples were there?

36. A freight train runs 6 miles an hour less than a passenger train. It runs 80 miles in the same time that the passenger train runs 112 miles. Find the rate of each train.

37. A and B each fire 40 times at a target; A's hits are one-half as numerous as B's misses, and A's misses exceed by 15 the number of B's hits. How many times does each hit the target?

38. A freight train travels from A to B at the rate of 12 miles an hour. After it has been gone $3\frac{1}{2}$ hours, an express train leaves A for B, travelling at the rate of 45 miles an hour, and reaches B 1 hour and 5 minutes ahead of the freight. Find the distance from A to B, and the time taken by the express train.

39. A tank has three taps. By the first it can be filled in 3 hours 10 minutes, by the second it can be filled in 4 hours 45 minutes, and by the third it can be emptied in 3 hours 48 minutes. How many hours will it take to fill it if all the taps are open?

40. A man invested a certain sum at $3\frac{3}{4}\%$, and $\frac{11}{15}$ this sum at $4\frac{1}{4}\%$; after paying an income tax of 5%, his net annual income is \$195.70. How much did he invest in each way?

VARIATION

156. One variable number (§ 124) is said to *vary directly* as another when the ratio of any two values of the first equals the ratio of the corresponding values of the second.

It is usual to omit the word "directly" and simply say that one number *varies* as another.

Thus, if a workman receives a fixed number of dollars per diem, the number of dollars received in m days will be to the number received in n days as m is to n.

Then, the ratio of any two numbers of dollars received equals the ratio of the corresponding numbers of days worked.

Hence, the number of dollars which the workman receives *varies* as the number of days during which he works.

157. The symbol $\propto$ is read "*varies as*"; thus, $a \propto b$ is read "a varies as b."

158. One variable number is said to *vary inversely* as another when the first varies directly as the *reciprocal* of the second.

Thus, the number of hours in which a railway train will traverse a fixed route varies inversely as the speed; if the speed be *doubled*, the train will traverse its route in *one-half* the number of hours.

159. One variable number is said to vary as two others *jointly* when it varies directly as their product.

Thus, the number of dollars received by a workman in a certain number of days varies jointly as the number which he receives in one day, and the number of days during which he works.

160. One variable number is said to vary directly as a second and inversely as a third, when it varies jointly as the second and the reciprocal of the third.

Thus, the attraction of a body varies directly as the amount of matter, and inversely as the square of the distance.

161. *If $x \propto y$, then x equals y multiplied by a constant number.*

Let x' and y' denote a *fixed* pair of corresponding values of x and y, and x and y any other pair.

By the definition of § 156, $\frac{x}{y} = \frac{x'}{y'}$; or, $x = \frac{x'}{y'}y$.

Denoting the constant ratio $\frac{x'}{y'}$ by m, we have

$$x = my.$$

162. It follows from §§ 158, 159, 160, and 161 that:

1. *If x varies inversely as y,* $x=\frac{m}{y}$.

2. *If x varies jointly as y and z,* $x=myz$.

3. *If x varies directly as y and inversely as z,* $x=\frac{my}{z}$.

163. *If $x \propto y$, and $y \propto z$, then $x \propto z$.*

By § 161, if $x \propto y$, $\qquad x=my. \qquad (1)$

And if $y \propto z$, $\qquad y=nz.$

Substituting in (1), $\qquad x=mnz.$

Whence, by § 161, $\qquad x \propto z.$

164. *If $x \propto y$ when z is constant, and $x \propto z$ when y is constant, then $x \propto yz$ when both y and z vary.*

Let y' and z' be the values of y and z, respectively, when x has the value x'.

Let y be changed from y' to y'', z remaining constantly equal to z', and let x be changed in consequence from x' to X.

Then, by § 156, $$\frac{x'}{X}=\frac{y'}{y''}. \qquad (1)$$

Now, let z be changed from z' to z'', y remaining constantly equal to y'', and let x be changed in consequence from X to x .

Then, $$\frac{X}{x''}=\frac{z'}{z''}. \qquad (2)$$

Multiplying (1) by (2), $$\frac{x'}{x''}=\frac{y'z'}{y''z''}. \qquad (3)$$

Now if *both* changes are made, that is, y from y' to y'' and z from z' to z'', x is changed from x' to x'', and yz is changed from $y'z'$ to $y''z''$.

Then by (3), the ratio of any two values of x equals the ratio of the *corresponding values* of yz; and, by § 156, $x \propto yz$.

The following is an illustration of the above theorem:

It is known, by Geometry, that the area of a triangle varies as the base when the altitude is constant, and as the altitude when the base is constant; hence, when both base and altitude vary, the area varies as their product.

165. Problems.

Problems in variation are readily solved by converting the variation into an equation by aid of §§ 161 or 162.

1. If x varies inversely as y, and equals 9 when $y=8$, find the value of x when $y=18$.

If x varies inversely as y, $x=\frac{m}{y}$ (§ 162).

Putting $x=9$ and $y=8$, $9=\frac{m}{8}$, or $m=72$.

Then, $x=\frac{72}{y}$; and, if $y=18$, $x=\frac{72}{18}=4$.

Since variation is simply another way of stating a proportion, the problems in variation may be solved readily by means of proportion.

E.g. In the above problem

$$x \propto \frac{1}{y},$$

$$x=\frac{m}{y}.$$

This equation is true for any assigned values of the variables.

Then,
$$x_1=\frac{m}{y_1}, \tag{1}$$

$$x_2=\frac{m}{y_2}. \tag{2}$$

Dividing (1) by (2)
$$\frac{x_1}{x_2}=\frac{y_2}{y_1} \tag{3}$$

which is in the form of inverse proportion. Substituting the given values of x and y in (3), we have

$$\frac{9}{x_2}=\frac{18}{8},$$

whence
$$x_2=\frac{9\cdot 8}{18}=4.$$

2. Given that the area of a triangle varies jointly as its base and altitude, what will be the base of a triangle whose altitude is 12, equivalent to the sum of two triangles whose bases are 10 and 6, and altitudes 3 and 9, respectively?

Let B, H, and A denote the base, altitude, and area, respectively, of any triangle, and B' the base of the required triangle.

Since A varies jointly as B and H, $A=mBH$ (§ 162).

Therefore, the area of the first triangle is $m\times 10\times 3$, or $30\,m$, and the area of the second is $m\times 6\times 9$, or $54\,m$.

Then, the area of the required triangle is $30\,m + 54\,m$, or $84\,m$.

But, the area of the required triangle is also $m \times B' \times 12$.

Therefore, $12\,mB' = 84\,m$, or $B' = 7$.

Or using proportion and letting A_1 = area of first triangle, A_2 = area of second, A_3 = area of third.

$$A_3 = A_1 + A_2$$

$$A_1 = mB_1H_1. \qquad (1)$$

$$A_2 = mB_2H_2. \qquad (2)$$

$$A_3 = mB_3H_3. \qquad (3)$$

Adding (1) and (2)

$$A_1 + A_2 = m(B_1H_1 + B_2H_2). \qquad (4)$$

Dividing (4) by (3)

$$\frac{A_1 + A_2}{A_3} = \frac{m(B_1H_1 + B_2H_2)}{m(B_3H_3)},$$

or,

$$1 = \frac{B_1H_1 + B_2H_2}{B_3H_3}. \qquad (5)$$

Substituting the given values of B and H in (5) we have

$$1 = \frac{10 \cdot 3 + 6 \cdot 9}{12\,B_3},$$

whence,

$$B_3 = 7.$$

EXERCISE 31

1. If $x \propto y$, and $x = 3$ when $y = 12$, what is the value of x when $y = 28$?

2. If $y \propto x^2$, and $y = 4$ when $x = 1$, what is the value of y in terms of x^2?

3. If y varies inversely as x, and $y = 4$ when $x = -3$, what is the value of y when $x = 2$?

4. If x varies directly as y and inversely as z, and $x = \frac{1}{2}$ when $y = \frac{2}{3}$ and $z = \frac{3}{4}$, what is the value of x when $y = \frac{4}{5}$ and $z = \frac{5}{12}$?

5. If x varies jointly as y and z and $x = -20$ when $y = 2$ and $z = 8$, what is the value of x when $y = -\frac{1}{2}$ and $z = 16$?

6. If $(3\,x + 4) \propto (2\,y - 5)$ when $x = -1$ and $y = 4$, what is the value of x when $y = 19$?

7. If x^2 varies inversely as y^3, when $x=4$ and $y=2$, what is the value of y when $x=\frac{4}{27}$?

8. If x equals the sum of two numbers, one of which varies directly as y and the other inversely as z^2, and $x=47$ when $y=-16$ and $z=2$, and $x=2$ when $y=-2$ and $z=1$, find the value of x when $y=3$ and $z=\frac{1}{3}$.

9. The area of a triangle varies jointly as its base and altitude. If the area of a triangle whose base is 6 and whose altitude is 9 is 27, what is the base of a triangle whose area is 44 and whose altitude is 11?

10. The distance through which a body falls from rest varies as the square of the time during which it falls. If a body falls 900 feet in 7.5 seconds, how many feet will it fall in 16 seconds?

11. The illumination from a source of light varies inversely as the square of the distance from the source. How far must an object 20 feet from the light be moved in order that it may receive twice as much light?

12. A circular plate of lead, 17 inches in diameter, is melted and formed into three circular plates of the same thickness. If the diameters of two of the plates are 8 and 9 inches respectively, find the diameter of the other; it being given that the area of a circle varies as the square of its diameter.

13. A cow tied to a stake by a rope 24 yards long will graze over the area within her reach in three days. She breaks her rope and, in repairing it, it is shortened $1\frac{1}{2}$ feet. In how many days will she graze over the new area?

14. A pump supplying the water for a building has a 10-inch stroke and a cylinder 4 inches in diameter. It is not possible to increase the number of strokes of the pump, nor to increase the length of the cylinder. By how much must the diameter be increased if 50% is added to the capacity of the pump? (The volumes of cylinders vary as the product of the base and altitude.)

VI. INVOLUTION AND EVOLUTION

166. We have already given (Chapter III) the involution and evolution of monomials. We will now consider involution and evolution of polynomials.

167. Square of a polynomial. By actual multiplication

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2\,ab + 2\,ac + 2\,bc.$$

In like manner

$$(a+b+c+d)^2$$
$$= a^2 + b^2 + c^2 + d^2 + 2\,ab + 2\,ac + 2\,ad + 2\,bc + 2\,bd + 2\,cd,$$

and so on for the square of any polynomial.

The law observed may be stated as follows:

The square of a polynomial is equal to the sum of the squares of its terms, together with twice the product of each term by each of the following terms.

Ex. Expand $(2\,x^2 - 3\,x - 5)^2$.

The squares of the terms are $4\,x^4$, $9\,x^2$, and 25.

Twice the product of the first term by each of the following terms gives the results $-12\,x^3$ and $-20\,x^2$.

Twice the product of the second term by the following term gives the result $30\,x$.

Then, $(2\,x^2 - 3\,x - 5)^2 = 4\,x^4 + 9\,x^2 + 25 - 12\,x^3 - 20\,x^2 + 30\,x$
$= 4\,x^4 - 12\,x^3 - 11\,x^2 + 30\,x + 25.$

168. Cube of a binomial. By actual multiplication

$$(a+b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3.$$

That is, **the cube of the sum of two numbers is equal to the cube of the first, plus three times the square of the first times the second, plus three times the first times the square of the second, plus the cube of the second.**

In like manner, the cube of the difference of two numbers is equal to the cube of the first, minus three times the square of the first times the second, plus three times the first times the square of the second, minus the cube of the second.

The cube of a *trinomial* may be found by the above method, if two of its terms be enclosed in parenthesis, and regarded as a single term.

169. Square Root of any Polynomial Perfect Square.

By § 167, $(a + b + c)^2 = a^2 + 2\,ab + b^2 + 2\,ac + 2\,bc + c^2$

$$= a^2 + (2\,a + b)\,b + (2\,a + 2\,b + c)c. \qquad (1)$$

Then, if the square of a trinomial be arranged in order of powers of some letter:

I. The square root of the first term gives the first term of the root, a.

II. If from (1) we subtract a^2, we have

$$(2\,a + b)\,b + (2\,a + 2\,b + c)\,c. \qquad (2)$$

The first term of this, when expanded, is $2\,ab$; if this be divided by twice the first term of the root, $2\,a$, we have the next term of the root, b.

III. If from (2) we subtract $(2\,a + b)b$, we have

$$(2\,a + 2\,b + c)\,c. \qquad (3)$$

The first term of this, when expanded, is $2\,ac$; if this be divided by twice the first term of the root, $2\,a$, we have the last term of the root, c.

IV. If from (3) we subtract $(2\,a + 2\,b + c)\,c$, there is no remainder.

Similar considerations hold with respect to the square of a polynomial of any number of terms.

170. The principles of § 169 may be used to find the square root of a polynomial perfect square of any number of terms.

Let it be required to find the square root of

$$4\,x^4 + 12\,x^3 - 7\,x^2 - 24\,x + 16.$$

$$\begin{array}{r|l|l}
 & 4\,x^4 + 12\,x^3 - 7\,x^2 - 24\,x + 16 & \underline{2\,x^2 + 3\,x - 4} \\
a^2 = 4\,x^4 & & \\
\hline
2\,a + b = 4\,x^2 + 3\,x & 12\,x^3 - 7\,x^2 - 24\,x + 16, & \text{1st Rem.} \\
3\,x & 12\,x^3 + 9\,x^2 & \\
\hline
2\,a + 2\,b + c = 4\,x^2 + 6\,x - 4 & -16\,x^2 - 24\,x + 16, & \text{2d Rem.} \\
-4 & -16\,x^2 - 24\,x + 16 &
\end{array}$$

$2\,x^2 + 3\,x - 4$ is called the square root and $2\,a$ the first trial divisor.
$2\,a + b$ is the first complete divisor.

We then have the following rule for extracting the square root of a polynomial perfect square:

Arrange the expression according to the powers of some letter.

Extract the square root of the first term, write the result as the first term of the root, and subtract its square from the given expression, arranging the remainder in the same order of powers as the given expression.

Divide the first term of the remainder by twice the first term of the root, and add the quotient to the part of the root already found, and also to the trial divisor.

Multiply the complete divisor by the term of the root last obtained, and subtract the product from the remainder.

If other terms remain, proceed as before, doubling the part of the root already found for the next trial divisor.

171. Cube Root of any Polynomial Perfect Cube.

By § 168, $(a+b+c)^3 = [(a+b)+c]^3$

$$= (a+b)^3 + 3(a+b)^2c + 3(a+b)c^2 + c^3$$

$$= a^3 + 3a^2b + 3ab^2 + b^3 + 3(a+b)^2c + 3(a+b)c^2 + c^3$$

$$= a^3 + (3a^2 + 3ab + b^2)b + [3(a+b)^2 + 3(a+b)c + c^2]c. \quad (1)$$

Then, if the cube of a trinomial be arranged in order of powers of some letter:

I. The cube root of the first term gives the first term of the cube root, a.

II. If from (1) we subtract a^3, we have

$$(3a^2 + 3ab + b^2)b + [3(a+b)^2 + 3(a+b)c + c^2]c. \quad (2)$$

The first term of this, when expanded, is $3a^2b$; if this be divided by three times the square of the first term of the root, $3a^2$, we have the next term of the root, b.

III. If from (2) we subtract $(3a^2 + 3ab + b^2)b$, we have

$$[3(a+b)^2 + 3(a+b)c + c^2]c. \quad (3)$$

The first term of this, when expanded, is $3a^2c$; if this be divided by three times the square of the first term of the root, $3a^2$, we have the last term of the root, c.

IV. If from (3) we subtract $[3(a+b)^2+3(a+b)c+c^2]c$, there is no remainder.

Similar considerations hold with respect to the cube of polynomials of any number of terms.

172. The principles of § 171 may be used to find the cube root of a polynomial perfect cube of any number of terms.

Let it be required to find the cube root of

$$x^6+6x^5+3x^4-28x^3-9x^2+54x-27.$$

$$\begin{array}{rr|l}
 & & \underline{x^2+2x-3} \\
 & & x^6+6x^5+3x^4-28x^3-9x^2+54x-27 \\
 & a^3=x^6 & \\
\hline
3a^2+3ab+b^2= & 3x^4+6x^3+4x^2 & 6x^5+3x^4-28x^3-9x^2+54x-27 \\
 & 2x & 6x^5+12x^4+8x^3 \\
\hline
3(a+b)^2= & 3x^4+12x^3+12x^2 & -9x^4-36x^3-9x^2+54x-27 \\
3(a+b)c+c^2= & -9x^2-18x+9 & \\
\cline{2-2}
 & 3x^4+12x^3+3x^2-18x+9 & \\
 & -3 & \underline{-9x^4-36x^3-9x^2+54x-27}
\end{array}$$

The first term of the root is the cube root of x^6, or x^2.

Subtracting the cube of x^2, or x^6, from the given expression, the first remainder is $6x^5+3x^4-28x^3-9x^2+54x-27$.

Dividing the first term of this by three times the square of the first term of the root, $3x^4$, we have the next term of the root, $2x$ (§ 171, II).

Now, $3ab+b^2$ equals $3\times x^2\times 2x+(2x)^2$, or $6x^3+4x^2$.

Adding this to $3x^4$, multiplying the result by $2x$, and subtracting the product, $6x^5+12x^4+8x^3$, from the first remainder, gives the second remainder, $-9x^4-36x^3-9x^2+54x-27$ (§ 171, III).

Dividing the first term of this by three times the square of the first term of the root, $3x^2$, we have the last term of the root, -3.

Now, $3(a+b)^2$ equals $3(x^2+2x)^2$, or $3x^4+12x^3+12x^2$; $3(a+b)c$ equals $3(x^2+2x)(-3)$, or $-9x^2-18x$; and $c^2=9$.

Adding these results, we have $3x^4+12x^3+3x^2-18x+9$.

Subtracting from the second remainder the product of this by -3, or $-9x^4-36x^3-9x^2+54x-27$, there is no remainder; then, x^2+2x-3 is the required root (§ 171, IV).

The expressions $3x^4$ and $3x^4+12x^3+12x^2$ are called ***trial divisors***, and the expressions $3x^4+6x^3+4x^2$ and $3x^4+12x^3+3x^2-18x+9$ ***complete divisors***.

We then have the following rule for finding the cube root of a polynomial perfect cube:

Arrange the expression according to the powers of some letter.

Extract the cube root of the first term, write the result as the first term of the root, and subtract its cube from the given expression; arranging the remainder in the same order of powers as the given expression.

Divide the first term of the remainder by three times the square of the first term of the root, and write the result as the next term of the root.

Add to the trial divisor three times the product of the term of the root last obtained by the part of the root previously found, and the square of the term of the root last obtained.

Multiply the complete divisor by the term of the root last obtained, and subtract the product from the remainder.

If other terms remain, proceed as before, taking three times the square of the *part of the root already found* for the next trial divisor.

EXERCISE 32

Find the square roots of the following:

1. $4a^4 + 12a^3b - 7a^2b^2 - 24ab^3 + 16b^4.$

2. $49m^4 - 5m^2 - 42m^3 + 1 + 6m.$

3. $9a^2 - 24ab - 36ac + 16b^2 + 48bc + 36c^2.$

4. $\frac{c^4}{4} + \frac{13c^2}{3} - 2c^3 + \frac{1}{9} - \frac{4c}{3}.$

5. $x^6 + 5x^4 + 14x^3 - 6x^5 + 1 - 4x - 2x^2.$

6. $4m + 25m^{\frac{1}{2}} - 12m^{\frac{3}{4}} + 16 - 24m^{\frac{1}{4}}.$

7. $64c^2 - 80c - 23 + 9c^{-2} + 30c^{-1}.$

8. $4x + 9y^{-4} + 4x^{\frac{1}{2}}y^{-2} + 24x^{\frac{1}{4}}y^{-3} - 16x^{\frac{3}{4}}y^{-1}.$

9. $6yz^{-2} + 4x^{-2} + y^2 - 4x^{-1}y - 12x^{-1}z^{-2} + 9z^{-4}.$

10. $4a^3 + 29a - 4a^{\frac{5}{2}} + 21a^2 - 20a^{\frac{1}{2}} + 4 - 18a^{\frac{3}{2}}$.

Find the cube roots of the following:

11. $343x^3 - 441x^2y + 189xy^2 - 27y^3$.

12. $x^6 - 9x^5 + 21x^4 + 9x^3 - 42x^2 - 36x - 8$.

13. $18a^4 - 13a^3 + 1 + 8a^6 + 9a^2 - 3a - 12a^5$.

14. $54m^5 + 44m^3 + 1 + 27m^6 + 63m^4 + 6m + 21m^2$.

15. $n^6 + 2n^4 + n^3 - \dfrac{2}{3}n^2 - 3n^5 - \dfrac{n}{3} - \dfrac{1}{27}$.

16. $64a^{\frac{3}{2}}b^{-6} - 240ab^{-4}c + 300a^{\frac{1}{2}}b^{-2}c^2 - 125c^3$.

17. $8s^3 + 36s^2 + 18s - 81 - 27s^{-1} + 81s^{-2} - 27s^{-3}$.

18. $21a^{\frac{1}{2}} - 54a^{\frac{5}{4}} + 27a^{\frac{3}{2}} + 63a - 44a^{\frac{3}{4}} + 1 - 6a^{\frac{1}{4}}$.

19. $x^{-3} - 3x^{-2}y^{\frac{1}{2}} + 3x^{-1}y - z^3 - 3x^{-2}z - y^{\frac{3}{2}} + 6x^{-1}y^{\frac{1}{2}}z - 3yz$
$+ 3x^{-1}z^2 - 3y\,z^2$.

20. $a + 6a^{\frac{2}{3}}b^{-1} + 12a^{\frac{1}{3}}b^{-2} + 8b^{-3} + 3a^{\frac{2}{3}}c^{-2} + 12a^{\frac{1}{3}}b^{-1}c^{-2}$
$+ 12b^{-2}c^{-2} + 3a^{\frac{1}{3}}c^{-4} + 6b^{-1}c^{-4} + c^{-6}$.

Find the fourth roots of the following:

21. $81a^{10} - 36a^{\frac{15}{2}}x^{-\frac{5}{3}} + 6a^5x^{-\frac{10}{3}} - \frac{4}{9}a^{\frac{5}{2}}x^{-5} + \frac{1}{81}x^{-\frac{20}{3}}$.

22. $x^8 - 12x^7 + 50x^6 - 72x^5 - 21x^4 + 72x^3 + 50x^2 + 12x + 1$.

Find the sixth roots of the following:

23. $64m^{12} - 192m^{10} + 240m^8 - 160m^6 + 60m^4 - 12m^2 + 1$.

24. $a^3 - 3a^{\frac{5}{2}}b^{\frac{3}{2}} + \frac{15}{4}a^2b^3 - \frac{5}{2}a^{\frac{3}{2}}b^{\frac{9}{2}} + \frac{15}{16}ab^6 - \frac{3}{16}a^{\frac{1}{2}}b^{\frac{15}{2}} + \frac{1}{64}b^9$.

173. Square Root of any Integral Perfect Square.

The square root of an integral perfect square may be found in the same way as the square root of a polynomial.

We have the following rule for finding the square root of an integral perfect square:

Separate the number into periods of two digits each, beginning with the units' place.

Find the greatest square in the left-hand period, and write its square root as the first digit of the root; subtract the square of the first root digit from the left-hand period, and to the result annex the next period.

Divide this remainder, omitting the last digit, by twice the part of the root already found, and annex the quotient to the root, and also to the trial divisor.

Multiply the complete divisor by the root digit last obtained, and subtract the product from the remainder.

If other periods remain, proceed as before, doubling the part of the root already found for the next trial divisor.

Note 1: It sometimes happens that, on multiplying a complete divisor by the digit of the root last obtained, the product is greater than the remainder.

In such a case, the digit of the root last obtained is too great, and one less must be substituted for it.

Note 2: If any root digit is 0, annex 0 to the trial divisor, and annex to the remainder the next period.

Ex. Required the square root of 15376.

$$
\begin{array}{rr|l}
 & 1'53'76 & 100+20+4 \\
 & a^2 = \underline{1\ 00\ 00} & = a+b+c \\
2a+b = 200+20 & 53\ 76 & \\
b = 20 & \underline{44\ 00} & = (2a+b)b \\
2a+2b+c = 200+40+4 & 9\ 76 & \\
4 & \underline{9\ 76} & = (2a+2b+c)c
\end{array}
$$

Pointing the number in accordance with the rule of § 173, we find that there are three digits in its square root.

Let a represent the hundreds' digit of the root, with two ciphers annexed; b the tens' digit, with one cipher annexed; and c the units' digit.

Then, a must be the greatest multiple of 100 whose square is less than 15376; this we find to be 100.

Subtracting a^2, or 10000, from the given number, the result is 5376.

Dividing the remainder by $2a$, or 200, we have the quotient 26+; which suggests that b equals 20.

Adding this to $2a$, or 200, and multiplying the result by b, or 20, we have 4400; which, subtracted from 5376, leaves 976.

Since this remainder equals $(2a + 2b + c)c$, we can get c approximately by dividing it by $2a + 2b$, or $200 + 40$.

Dividing 976 by 240, we have the quotient 4+; which suggests that c equals 4.

Adding this to 240, multiplying the result by 4, and subtracting the product, 976, there is no remainder.

Then 124 is the square root.

Omitting the ciphers for the sake of brevity, and condensing the operation, we may arrange the work of the example as follows:

$$
\begin{array}{r|l}
 & 1'53'76 \,\underline{|\,124} \\
 & 1 \\
\hline
22 & 53 \\
 & 44 \\
\hline
244 & 976 \\
 & 976 \\
\end{array}
$$

CUBE ROOT OF AN ARITHMETICAL NUMBER

174. The cube root of 1000 is 10; of 1000000 is 100, etc.

Hence, the cube root of a number between 1 and 1000 is between 1 and 10; the cube root of a number between 1000 and 1000000 is between 10 and 100; etc.

That is, the integral part of the cube root of an integer of one, two, or three digits contains *one* digit; of an integer of four, five, or six digits contains *two* digits; and so on.

Hence, if a point be placed over every third digit of an integer, beginning at the units' place, the number of points shows the number of digits in the integral part of its cube root.

175. Cube Root of any Integral Perfect Cube.

The cube root of an integral perfect cube may be found in the same way as the cube root of a polynomial.

Required the cube root of 12487168.

$$
\begin{array}{rr|l|l}
 & & \dot{1}248\dot{7}16\dot{8} & 200 + 30 + 2 \\
 & & a^3 = 8000000 & = a + b + c \\
\cline{3-3}
3\,a^2 = & 120000 & 4487168 & \\
3\,ab = & 18000 & & \\
b^2 = & 900 & & \\
\cline{2-2}
 & 138900 & & \\
 & 30 & 4167000 & \\
\cline{2-3}
3(a+b)^2 = & 158700 & 320168 & \\
3(a+b)c = & 1380 & & \\
c^2 = & 4 & & \\
\cline{2-2}
 & 160084 & & \\
 & 2 & 320168 & \\
\cline{3-3}
\end{array}
$$

Pointing the number in accordance with the rule of § 174, we find that there are three digits in the cube root.

Let a represent the hundreds' digit of the root, with two ciphers annexed; b the tens' digit, with one cipher annexed; and c the units' digit.

Then, a must be the greatest multiple of 100 whose cube is less than 12487168; this we find to be 200.

Subtracting a^3, or 8000000, from the given number, the result is 4487168.

Dividing this by $3\,a^2$, or 120000, we have the quotient 37+; which suggests that b equals 30.

Adding to the divisor 120000, $3\,ab$, or 18000, and b^2, or 900, we have 138900.

Multiplying this by b, or 30, and subtracting the product 4167000 from 4487168, we have 320168.

Since this remainder equals $[3(a+b)^2 + 3(a+b)c + c^2]c$ (§ 171, III), we can get c approximately by dividing it by $3(a+b)^2$, or 158700.

Dividing 320168 by 158700, the quotient is 2+; which suggests that c equals 2.

Adding to the divisor 158700, $3(a+b)c$, or 1380, and c^2, or 4, we have 160084; multiplying this by 2, and subtracting the product, 320168, there is no remainder.

Then, $200 + 30 + 2$, or 232, is the required cube root.

176. Omitting the ciphers for the sake of brevity, and condensing the process, the work of the example of § 175 will stand as follows:

	12487168	282
	8	
1200	4487	
180		
9		
1389	4167	
158700	320168	
1380		
4		
160084	320168	

The numbers 120000 and 158700 are called *trial divisors*, and the numbers 138900 and 160084 are called *complete divisors.*

We then have the following rule for finding the cube root of an integral perfect cube:

Separate the number into periods by pointing every third digit, beginning with the units' place.

Find the greatest cube in the left-hand period, and write its cube root as the first digit of the root; subtract the cube of the first root digit from the left-hand period, and to the result annex the next period.

Divide this remainder by three times the square of the part of the root already found, with two ciphers annexed, and write the quotient as the next digit of the root.

Add to the trial divisor three times the product of the last root digit by the part of the root previously found, with one cipher annexed, and the square of the last root digit.

Multiply the complete divisor by the digit of the root last obtained, and subtract the product from the remainder.

If other periods remain, proceed as before, taking three times the square of the part of the root already found, with two ciphers annexed, for the next trial divisor.

Note 1: Note 1, § 173, applies with equal force to the above rule.

Note 2: If any root-figure is 0, annex two ciphers to the trial divisor, and annex to the remainder the next period.

177. In the example of § 175, the first complete divisor is

$$3\,a^2 + 3\,ab + b^2. \qquad (1)$$

The next trial divisor is $3\,(a + b)^2$, or $3\,a^2 + 6\,ab + 3\,b^2$.

This may be obtained from (1) by adding to it its second term, and double its third term.

That is, **if the first number and the double of the second number required to complete any trial divisor be added to the complete divisor, the result, with two ciphers annexed, will give the next trial divisor.**

This rule saves much labor in forming the trial divisors.

Ex. Find the cube root of 157464.

$$\begin{array}{r|l}
 & \dot{1}5\dot{7}46\dot{4} \;\underline{|\,54} \\
 & 125 \\
\hline
7500 & 32464 \\
600 & \\
16 & \\
\hline
8116 & \underline{32464}
\end{array}$$

EXERCISE 33

Find the square roots of the following:

1. 5294601.
2. 68.7241.
3. .00098596.
4. 567762.25.
5. .0037319881.

Find the cube roots of the following:

6. 658503.
7. 1953125.
8. 748.613312.
9. .000070444997.
10. .000001601613.

Find the first four figures of the square roots of:

11. 3.
12. $\frac{7}{8}$.
13. $\frac{15}{16}$.
14. $\frac{2}{7}$.
15. $\frac{2075}{3072}$.

Find the first four figures of the cube roots of:

16. 5.
17. 16.
18. $\frac{1}{4}$.
19. .27.
20. $\frac{3}{40}$.

OTHER POWERS

178. A **Series** is a succession of terms.

A **Finite Series** is one having a limited number of terms.

An **Infinite Series** is one having an unlimited number of terms.

179. In §§ 103 and 168 we gave rules for finding the square or cube of any binomial.

The **Binomial Theorem** is a formula by means of which any power of a binomial may be expanded into a series.

180. Proof of the Binomial Theorem for a Positive Integral Exponent.

The following are obtained by actual multiplication:

$$(a+x)^2 = a^2 + 2\,ax + x^2;$$

$$(a+x)^3 = a^3 + 3\,a^2x + 3\,ax^2 + x^3;$$

$$(a+x)^4 = a^4 + 4\,a^3x + 6\,a^2x^2 + 4\,ax^3 + x^4; \text{ etc.}$$

In these results, we observe the following laws:

1. The number of terms is greater by 1 than the exponent of the binomial.

2. The exponent of a in the first term is the same as the exponent of the binomial, and decreases by 1 in each succeeding term.

3. The exponent of x in the second term is 1, and increases by 1 in each succeeding term.

4. The coefficient of the first term is 1, and the coefficient of the second term is the exponent of the binomial.

5. If the coefficient of any term be multiplied by the exponent of a in that term, and the result divided by the exponent of x in the term increased by 1, the quotient will be the coefficient of the next following term.

181. If the laws of § 180 be assumed to hold for the expansion of $(a+x)^n$, where n is any positive integer, the exponent of a in the first term is n, in the second term $n-1$, in the third term $n-2$, in the fourth term $n-3$, etc.

The exponent of x in the second term is 1, in the third term 2, in the fourth term 3, etc.

The coefficient of the first term is 1; of the second term n.

Multiplying the coefficient of the second term, n, by $n-1$, the exponent of a in that term, and dividing the result by the exponent of x in the term increased by 1, or 2, we have $\frac{n(n-1)}{1\cdot 2}$ as the coefficient of the third term; and so on.

Then,
$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1\cdot 2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}x^3 + \cdots. \qquad (1)$$

Multiplying both members of (1) by $a+x$, we have

$$\begin{aligned}(a+x)^{n+1} = a^{n+1} + na^n x &+ \frac{n(n-1)}{1\cdot 2}a^{n-1}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-2}x^3 + \cdots\\ + a^n x &+ \qquad na^{n-1}x^2 + \qquad\quad \frac{n(n-1)}{1\cdot 2}a^{n-2}x^3 + \cdots.\end{aligned}$$

Collecting the terms which contain like powers of a and x, we have

$$\begin{aligned}(a+x)^{n+1} &= a^{n+1} + (n+1)a^n x + \left[\frac{n(n-1)}{1\cdot 2} + n\right]a^{n-1}x^2\\ &\quad + \left[\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} + \frac{n(n-1)}{1\cdot 2}\right]a^{n-2}x^3 + \cdots\\ &= a^{n+1} + (n+1)a^n x + n\left[\frac{n-1}{2} + 1\right]a^{n-1}x^2\\ &\quad + \frac{n(n-1)}{1\cdot 2}\left[\frac{n-2}{3} + 1\right]a^{n-2}x^3 + \cdots.\end{aligned}$$

Then,
$$\begin{aligned}(a+x)^{n+1} &= a^{n+1} + (n+1)a^n x + n\left[\frac{n+1}{2}\right]a^{n-1}x^2\\ &\quad + \frac{n(n-1)}{1\cdot 2}\left[\frac{n+1}{3}\right]a^{n-2}x^3 + \cdots\\ &= a^{n+1} + (n+1)a^n x + \frac{(n+1)n}{1\cdot 2}a^{n-1}x^2\\ &\quad + \frac{(n+1)n(n-1)}{1\cdot 2\cdot 3}a^{n-2}x^3 + \cdots. \qquad (2)\end{aligned}$$

It will be observed that this result is in accordance with the laws of § 180; which proves that, if the laws hold for any power of $a+x$ whose exponent is a positive integer, they also hold for a power whose exponent is greater by 1.

But the laws have been shown to hold for $(a+x)^4$, and hence they also hold for $(a+x)^5$; and since they hold for $(a+x)^5$, they also hold for $(a+x)^6$; and so on.

Therefore, the laws hold when the exponent is any positive integer, and equation (1) is proved for every positive integral value of n.

Equation (1) is called the *Binomial Theorem.*

In place of the denominators $1 \cdot 2$; $1 \cdot 2 \cdot 3$, etc., it is usual to write $\underline{|2}$, $\underline{|3}$, etc.

The symbol $\underline{|n}$, read "factorial n," signifies the product of the natural numbers from 1 to n, inclusive.

The method of proof exemplified in § 181 is known as *Mathematical Induction.*

182. Putting $a=1$ in equation (1), § 181, we have

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{\underline{|2}}x^2 + \frac{n(n-1)(n-2)}{\underline{|3}}x^3 + \cdots.$$

183. In expanding expressions by the Binomial Theorem, it is convenient to obtain the exponents and coefficients of the terms by aid of the laws of § 180.

1. Expand $(a+x)^5$.

The exponent of a in the first term is 5, and decreases by 1 in each succeeding term.

The exponent of x in the second term is 1, and increases by 1 in each succeeding term.

The coefficient of the first term is 1; of the second, 5.

Multiplying 5, the coefficient of the second term, by 4, the exponent of a in that term, and dividing the result by the exponent of x increased by 1, or 2, we have 10 as the coefficient of the third term; and so on.

Then, $(a+x)^5 = a^5 + 5a^4x + 10a^3x + 10a^2x^3 + 5ax^4 + x^5$.

It will be observed that the coefficients of terms equally distant from the ends of the expansion are equal; this law will be proved in § 185.

Thus the coefficients of the latter half of an expansion may be written out from the first half.

If the second term of the binomial is *negative*, it should be enclosed, negative sign and all, in parentheses before applying the laws.

2. Expand $(1-x)^6$.

$$\begin{aligned}(1-x)^6 &= [1+(-x)]^6\\ &= 1^6+6\cdot 1^5\cdot(-x)+15\cdot 1^4\cdot(-x)^2+20\cdot 1^3\cdot(-x)^3\\ &\quad +15\cdot 1^2\cdot(-x)^4+6\cdot 1\cdot(-x)^5+(-x)^6\\ &= 1-6x+15x^2-20x^3+15x^4-6x^5+x^6.\end{aligned}$$

If the first term of the binomial is an arithmetical number, it is convenient to write the exponents at first without reduction; the result should afterwards be reduced to its simplest form.

If either term of the binomial has a coefficient or exponent other than unity, it should be enclosed in parentheses before applying the laws.

3. Expand $(3m^2-\sqrt[3]{n})^4$.

$$\begin{aligned}(3m^2-\sqrt[3]{n})^4 &= [(3m^2)+(-n^{\frac{1}{3}})]^4\\ &= (3m^2)^4+4(3m^2)^3(-n^{\frac{1}{3}})+6(3m^2)^2(-n^{\frac{1}{3}})^2\\ &\quad +4(3m^2)(-n^{\frac{1}{3}})^3+(-n^{\frac{1}{3}})^4\\ &= 81m^8-108m^6n^{\frac{1}{3}}+54m^4n^{\frac{2}{3}}-12m^2n+n^{\frac{4}{3}}.\end{aligned}$$

A *trinomial* may be raised to any power by the Binomial Theorem, if two of its terms be enclosed in parentheses, and regarded as a single term; but for second powers, the method of § 167 is shorter.

4. Expand $(x^2-2x-2)^4$.

$$\begin{aligned}(x^2-2x-2)^4 &= [(x^2-2x)+(-2)]^4\\ &= (x^2-2x)^4+4(x^2-2x)^3(-2)+6(x^2-2x)^2(-2)^2\\ &\quad +4(x^2-2x)(-2)^3+(-2)^4\\ &= x^8-8x^7+24x^6-32x^5+16x^4\\ &\quad -8(x^6-6x^5+12x^4-8x^3)\\ &\quad +24(x^4-4x^3+4x^2)-32(x^2-2x)+16\\ &= x^8-8x^7+16x^6+16x^5-56x^4-32x^3+64x^2+64x+16\end{aligned}$$

EXERCISE 34

Expand the following:

1. $(a+b)^6$.
2. $(x-y)^9$.
3. $(1-x)^7$.
4. $(xy+z)^8$.
5. $(a^2-b)^5$.
6. $(2a-b)^5$.
7. $(3m-4n)^4$.
8. $(p^{\frac{1}{2}}-2q)^6$.
9. $(x^{-2}+y^{\frac{3}{2}})^5$
10. $(2a^{-\frac{1}{5}}+b^{\frac{1}{5}})^7$.
11. $\left(\dfrac{x^2}{2}-3y^{-1}\right)^6$.
12. $\left(\dfrac{a^2}{b}-\dfrac{b^2}{a}\right)^8$.
13. $\left(\dfrac{m}{3n}+3\sqrt{mn}\right)^4$.
14. $(2c^{-\frac{2}{3}}-\frac{1}{5}d^{-\frac{4}{5}})^5$.
15. $(y-z^2)^{10}$.
16. $(1-a^2)^{12}$.
17. $(\sqrt[4]{a^3}-\frac{1}{4}\sqrt[3]{a^2})^4$.
18. $(a+b)^{15}$.
19. $\left(\dfrac{1}{2}a^{-\frac{1}{3}}+\dfrac{3}{4a^{-\frac{2}{3}}}\right)^5$.
20. $(x^{\frac{5}{3}}-y^4z^3)^{11}$.
21. $(a+b-c)^4$.
22. $(x^2-2x-3)^4$.
23. $(m^2-2m+1)^4$.
24. $(x^2+x+1)^5$.
25. $(1+c+c^2)^6$.

184. *To find the rth or general term in the expansion of* $(a+x)^n$.

The following laws hold for any term in the expansion of $(a+x)^n$, in equation (1), § 181:

1. The exponent of x is less by 1 than the number of the term.

2. The exponent of a is n minus the exponent of x.

3. The last factor of the numerator is greater by 1 than the exponent of a.

4. The last factor of the denominator is the same as the exponent of x.

Therefore in the rth term, the exponent of x will be $r-1$.
The exponent of a will be $n-(r-1)$, or $n-r+1$.
The last factor of the numerator will be $n-r+2$.
The last factor of the denominator will be $r-1$.

Hence, the rth term

$$=\frac{n(n-1)(n-2)\cdots(n-r+2)}{1\cdot 2\cdot 3\cdots(r-1)}a^{n-r+1}x^{r-1}. \qquad (1)$$

In finding any term of an expansion, it is convenient to obtain the coefficient and exponents of the terms by the above laws.

Ex. Find the 8th term of $(3a^{\frac{1}{2}}-b^{-1})^{11}$.

We have, $(3a^{\frac{1}{2}}-b^{-1})^{11}=[(3a^{\frac{1}{2}})+(-b^{-1})]^{11}$.
In this case, $n=11$, $r=8$.
The exponent of $(-b^{-1})$ is $8-1$, or 7.
The exponent of $(3a^{\frac{1}{2}})$ is $11-7$, or 4.
The first factor of the numerator is 11, and the last factor $4+1$, or 5.
The last factor of the denominator is 7.

$$\text{Then, the 8th term}=\frac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}(3a^{\frac{1}{2}})^4(-b^{-1})^7$$
$$=330(81a^2)(-b^{-7})=-26730a^2b^{-7}.$$

If the second term of the binomial is negative, it should be enclosed, sign and all, in parentheses before applying the laws.

If either term of the binomial has a coefficient or exponent other than unity, it should be enclosed in parentheses before applying the laws.

EXERCISE 35

Find the:

1. 5th term of $(a+b)^9$.
2. 7th term of $(x-y)^{10}$.
3. 6th term of $(1-x)^{11}$.
4. 4th term of $(a^2-b^3)^8$.
5. 8th term of $(c^{\frac{1}{2}}-2d^3)^{12}$.
6. 10th term of $\left(m^{\frac{2}{3}}+\frac{n}{2}\right)^{14}$.
7. 7th term of $\left(c^{-2}-\frac{1}{3x}\right)^{12}$.
8. 6th term of $\left(\frac{a^2}{b^2}+\frac{b}{a}\right)^8$.
9. 5th term of $\left(\sqrt{\frac{a}{3}}-\sqrt{\frac{a}{2}}\right)^9$.
10. 4th term of $(x\sqrt{y}-\frac{2}{3}y^{-\frac{1}{4}})^7$.

185. Multiplying both terms of the coefficient, in (1), § 184, by the product of the natural numbers from 1 to $n-r+1$, inclusive, the coefficient of the rth term becomes

$$\frac{n(n-1)\cdots(n-r+2)\cdot(n-r+1)\cdots 2\cdot 1}{\underline{|r-1}\times 1\cdot 2\cdots(n-r+1)}=\frac{\underline{|n}}{\underline{|r-1}\,\underline{|n-r+1}}.$$

Since the number of terms in the expansion is $n+1$, the rth term from the end is the $(n-r+2)$th from the beginning.

Then, to find the coefficient of the rth term from the end, we put in the above formula $n-r+2$ for r.

Then, the coefficient of the rth term from the end is

$$\frac{\lfloor n}{\lfloor n-r+2-1 \; \lfloor n-(n-r+2)+1}, \text{ or } \frac{\lfloor n}{\lfloor n-r+1 \; \lfloor r-1}.$$

Hence, in the expansion of $(a+x)^n$, the coefficients of terms equidistant from the ends of the expansion are equal.

186. It was proved in § 181 that, if n is a positive integer,

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1 \cdot 2}a^{n-2}x^2$$
$$+ \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}x^3 + \cdots.$$

If n is a negative integer, or a positive or negative fraction, the series in the second member is infinite (§ 178); for no one of the expressions $n-1$, $n-2$, etc., can equal zero; in this case, the series gives the value of $(a+x)^n$, provided it is convergent.

As a rigorous proof of the Binomial Theorem for Fractional and Negative Exponents is too difficult for pupils at this stage of their progress, the author has thought best to omit it; any one desiring a rigorous algebraic proof of the theorem, will find it in the author's Advanced Course in Algebra, § 575.

187. Examples.

In expanding expressions by the Binomial Theorem when the exponent is fractional or negative, the exponents and coefficients of the terms may be found by the laws of § 180, which hold for all values of the exponent.

1. Expand $(a+x)^{\frac{2}{3}}$ to five terms.

The exponent of a in the first term is $\frac{2}{3}$, and decreases by 1 in each succeeding term.

The exponent of x in the second term is 1, and increases by 1 in each succeeding term.

The coefficient of the first term is 1; of the second term, $\frac{2}{3}$.

Multiplying $\frac{2}{3}$, the coefficient of the second term, by $-\frac{1}{3}$, the exponent of a in that term, and dividing the product by the exponent of x increased by 1, or 2, we have $-\frac{1}{9}$ as the coefficient of the third term; and so on.

Then, $(a+x)^{\frac{2}{3}} = a^{\frac{2}{3}} + \frac{2}{3}a^{-\frac{1}{3}}x - \frac{1}{9}a^{-\frac{4}{3}}x^2 + \frac{4}{81}a^{-\frac{7}{3}}x^3 - \frac{7}{243}a^{-\frac{10}{3}}x^4 + \cdots.$

2. Expand $(1 + 2x^{-\frac{1}{2}})^{-2}$ to five terms.

Enclosing $2x^{-\frac{1}{2}}$ in parentheses, we have

$$\begin{aligned}(1+2x^{-\frac{1}{2}})^{-2} &= [1 + (2x^{-\frac{1}{2}})]^{-2}\\ &= 1^{-2} - 2\cdot 1^{-3}\cdot(2x^{-\frac{1}{2}}) + 3\cdot 1^{-4}\cdot(2x^{-\frac{1}{2}})^2\\ &\qquad - 4\cdot 1^{-5}\cdot(2x^{-\frac{1}{2}})^3 + 5\cdot 1^{-6}\cdot(2x^{-\frac{1}{2}})^4 - \cdots\\ &= 1 - 4x^{-\frac{1}{2}} + 12x^{-1} - 32x^{-\frac{3}{2}} + 80x^{-2} + \cdots.\end{aligned}$$

By writing the exponents of 1, in expanding $[1 + (2x^{-\frac{1}{2}})]^{-2}$, we can make use of the fifth law of § 180.

3. Expand $\dfrac{1}{\sqrt[3]{a^{-1} - 3x^{\frac{1}{3}}}}$ to four terms.

Enclosing a^{-1} and $-3x^{\frac{1}{3}}$ in parentheses, we have

$$\begin{aligned}\frac{1}{\sqrt[3]{a^{-1}-3x^{\frac{1}{3}}}} = \frac{1}{(a^{-1}-3x^{\frac{1}{3}})^{\frac{1}{3}}} &= [(a^{-1}) + (-3x^{\frac{1}{3}})]^{-\frac{1}{3}}\\ &= (a^{-1})^{-\frac{1}{3}} - \tfrac{1}{3}(a^{-1})^{-\frac{4}{3}}(-3x^{\frac{1}{3}}) + \tfrac{2}{9}(a^{-1})^{-\frac{7}{3}}(-3x^{\frac{1}{3}})^2\\ &\qquad - \tfrac{14}{81}(a^{-1})^{-\frac{10}{3}}(-3x^{\frac{1}{3}})^3 + \cdots\\ &= a^{\frac{1}{3}} + a^{\frac{4}{3}}x^{\frac{1}{3}} + 2a^{\frac{7}{3}}x^{\frac{2}{3}} + \tfrac{14}{3}a^{\frac{10}{3}}x + \cdots.\end{aligned}$$

EXERCISE 36

Expand each of the following to five terms:

1. $(a+x)^{\frac{3}{4}}$.
2. $(1+x)^{-8}$.
3. $(1-x)^{-\frac{3}{4}}$.
4. $\sqrt[5]{a-b}$.
5. $\dfrac{1}{(a+x)^5}$.
6. $\dfrac{1}{\sqrt[6]{1-x}}$.
7. $(a^{\frac{4}{3}} + 2b)^{\frac{3}{4}}$.
8. $(a^3 - 4x^{\frac{1}{2}})^{-\frac{7}{2}}$.
9. $\dfrac{1}{x^{-\frac{2}{3}} + 3y}$.

10. $\left(m^{-3}+\dfrac{n^{-\frac{5}{6}}}{4}\right)^{-3}$.

11. $\sqrt[3]{[(a^{-2}-6b^2c)^7]}$.

12. $\dfrac{1}{(x^{-\frac{1}{3}}-2y^{\frac{3}{2}})^4}$.

13. $\left(\dfrac{x^2}{y}+\dfrac{y^2}{x}\right)^{-\frac{1}{3}}$.

14. $(m^{\frac{5}{3}}-3n^{-\frac{4}{3}})^{-\frac{4}{3}}$.

15. $\left(\dfrac{1}{5\sqrt[5]{a^4}}-\sqrt[3]{b^2}\right)^{\frac{3}{5}}$.

188. The formula for the rth term of $(a+x)^n$ (§ 184) holds for fractional or negative values of n, since it was derived from an expansion which holds for all values of the exponent.

Ex. Find the 7th term of $(a-3x^{-\frac{3}{2}})^{-\frac{1}{3}}$.

Enclosing $-3x^{-\frac{3}{2}}$ in parentheses, we have

$$(a-3x^{-\frac{3}{2}})^{-\frac{1}{3}}=[a+(-3x^{-\frac{3}{2}})]^{-\frac{1}{3}}.$$

The exponent of $(-3x^{-\frac{3}{2}})$ is $7-1$, or 6.
The exponent of a is $-\frac{1}{3}-6$, or $-\frac{19}{3}$.
The first factor of the numerator is $-\frac{1}{3}$, and the last factor $-\frac{19}{3}+1$, or $-\frac{16}{3}$.
The last factor of the denominator is 6.
Hence, the 7th term

$$=\frac{\frac{1}{3}\cdot-\frac{4}{3}\cdot-\frac{7}{3}\cdot-\frac{10}{3}\cdot-\frac{10}{3}\cdot-\frac{16}{3}}{1\cdot2\cdot3\cdot4\cdot5\cdot6}a^{-\frac{19}{3}}(-3x^{-\frac{3}{2}})^6$$

$$=\frac{728}{3^8}a^{-\frac{19}{3}}(3^6x^{-9})=\frac{728}{9}a^{-\frac{19}{3}}x^{-9}.$$

EXERCISE 37

Find the:

1. 6th term of $(a+x)^{\frac{2}{3}}$.
2. 5th term of $(a-b)^{-\frac{1}{5}}$.
3. 7th term of $(1+x)^{-7}$.
4. 8th term of $(1-x)^{\frac{3}{5}}$.
5. 9th term of $(a-x)^{-3}$.
6. 11th term of $\sqrt{(m+n)^5}$.
7. 7th term of $(a^{-2}-2b^{\frac{1}{3}})^{-2}$.
8. 8th term of $\dfrac{1}{(x^5+y^{-\frac{1}{2}})^4}$.
9. 10th term of $(x^{-5}+y^{\frac{2}{3}})^{-\frac{5}{2}}$.
10. 6th term of $(a^{\frac{3}{2}}-2b^{-4})^{-\frac{3}{4}}$.
11. 5th term of $(m+3n^{-3})^{\frac{8}{3}}$.
12. 9th term of $\dfrac{1}{\sqrt[3]{[(a^3+3b^{-\frac{2}{3}})^5]}}$.

189. Extraction of Roots.

The Binomial Theorem may sometimes be used to find the approximate root of a number which is not a perfect power of the same degree as the index of the root.

Ex. Find $\sqrt[3]{25}$ approximately to five places of decimals.

The nearest perfect cube to 25 is 27.

We have $\sqrt[3]{25} = \sqrt[3]{27-2} = [(3^3)+(-2)]^{\frac{1}{3}}$

$$= (3^3)^{\frac{1}{3}} + \tfrac{1}{3}(3^3)^{-\frac{2}{3}}(-2) - \tfrac{1}{9}(3^3)^{-\frac{5}{3}}(-2)^2$$
$$+ \tfrac{5}{81}(3^3)^{-\frac{8}{3}}(-2)^3 - \cdots$$
$$= 3 - \frac{2}{3\cdot 3^2} - \frac{4}{9\cdot 3^5} - \frac{40}{81\cdot 3^8} - \cdots.$$

Expressing each fraction approximately to the nearest fifth decimal place, we have

$$\sqrt[3]{25} = 3 - .07407 - .00183 - .00008 - \cdots = 2.92402.$$

We then have the following rule:

Separate the given number into two parts, the first of which is the nearest perfect power of the same degree as the required root, and expand the result by the Binomial Theorem.

If the ratio of the second term of the binomial to the first is a small proper fraction, the terms of the expansion diminish rapidly; but if this ratio is but little less than 1, it requires a great many terms to insure any degree of accuracy.

EXERCISE 38

Find the approximate values of the following to five places of decimals:

1. $\sqrt{17}$. 2. $\sqrt{51}$. 3. $\sqrt[3]{60}$. 4. $\sqrt[4]{14}$. 5. $\sqrt[4]{84}$. 6. $\sqrt[5]{35}$.

PROPERTIES OF QUADRATIC SURDS

190. A quadratic surd (§ 70) cannot equal the sum of a rational expression and a quadratic surd.

For, if possible, let $(a)^{\frac{1}{2}} = b + (c)^{\frac{1}{2}}$,

where b is a rational expression, and $(a)^{\frac{1}{2}}$ and $(c)^{\frac{1}{2}}$ quadratic surds.

Squaring both members, $a = b^2 + 2b(c)^{\frac{1}{2}} + c,$

or, $2b(c)^{\frac{1}{2}} = a - b^2 - c.$

Whence, $(c)^{\frac{1}{2}} = \dfrac{a - b^2 - c}{2b}.$

That is, a quadratic surd equal to a rational expression.

But this is impossible; whence, $(a)^{\frac{1}{2}}$ cannot equal $b + (c)^{\frac{1}{2}}$.

191. If $a + (b)^{\frac{1}{2}} = c + (d)^{\frac{1}{2}}$, where a and c are rational expressions, and $(b)^{\frac{1}{2}}$ and $(d)^{\frac{1}{2}}$ quadratic surds, then

$$a = c, \text{ and } (b)^{\frac{1}{2}} = (d)^{\frac{1}{2}}.$$

If a does not equal c, let $a = c + x$; then, x is rational.

Substituting this value in the given equation,

$$c + x + (b)^{\frac{1}{2}} = c + (d)^{\frac{1}{2}}, \text{ or } x + (b)^{\frac{1}{2}} = (d)^{\frac{1}{2}}.$$

But this is impossible by § 190.

Then, $a = c$, and therefore $(b)^{\frac{1}{2}} = (d)^{\frac{1}{2}}$.

192. If $(a + \sqrt{b})^{\frac{1}{2}} = \sqrt{x} + \sqrt{y}$, where a, b, x, and y are rational expressions, then $(a - \sqrt{b})^{\frac{1}{2}} = \sqrt{x} - \sqrt{y}$.

Squaring both members of the given equation,

$$a + \sqrt{b} = x + 2\sqrt{xy} + y,$$

Whence, by § 191, $a = x + y,$

and $(b)^{\frac{1}{2}} = 2(xy)^{\frac{1}{2}}.$

Subtracting, $a - (b)^{\frac{1}{2}} = x - 2(xy)^{\frac{1}{2}} + y.$

Extracting the square root of both members,

$$(a - \sqrt{b})^{\frac{1}{2}} = \sqrt{x} - \sqrt{y}.$$

193. Square Root of a Binomial Surd.

The preceding principles may be used to find the square roots of certain expressions which are in the form of the sum of a rational expression and a quadratic surd.

Ex. Find the square root of $13-\sqrt{160}$.

Assume, $\sqrt{13-\sqrt{160}}=\sqrt{x}-\sqrt{y}$. (1)

Then, by § 192, $\sqrt{13+\sqrt{160}}=\sqrt{x}+\sqrt{y}$. (2)

Multiply (1) by (2), $\sqrt{169-160}=x-y$.

Or, $x-y=3$. (3)

Squaring (1) $13-\sqrt{160}=x-2\sqrt{xy}+y$.

Whence, by § 191, $x+y=13$. (4)

Adding (3) and (4), $2x=16$, or $x=8$.

Subtracting (3) from (4), $2y=10$, or $y=5$.

Substitute in (1), $\sqrt{13-\sqrt{160}}=\sqrt{8}-\sqrt{5}=2\sqrt{2}-\sqrt{5}$.

194. Examples like that of § 193 may be solved by inspection, by putting the given expression into the form of a trinomial perfect square (§ 103, II), as follows:

Reduce the surd term so that its coefficient may be 2.

Separate the rational term into two parts whose product shall be the expression under the radical sign of the surd term.

Extract the square root of each part, and connect the results by the sign of the surd term.

1. Extract the square root of $8+\sqrt{48}$.

We have $\sqrt{48}=2\sqrt{12}$.

We then separate 8 into two parts whose product is 12.

The parts are 6 and 2; whence,

$$\sqrt{8+\sqrt{48}}=\sqrt{6+2\sqrt{12}+2}=\sqrt{6}+\sqrt{2}.$$

2. Extract the square root of $22-3\sqrt{32}$.

We have $3\sqrt{32}=\sqrt{9\times 8\times 4}=2\sqrt{72}$.

We then separate 22 into two parts whose product is 72.

The parts are 18 and 4; whence,

$$\sqrt{22-3\sqrt{32}}=\sqrt{18-2\sqrt{72}+4}=\sqrt{18}-\sqrt{4}=3\sqrt{2}-2.$$

EXERCISE 39

Find the square roots of the following:

1. $5+2\sqrt{6}$.
2. $8-2\sqrt{12}$.
3. $8a-2a\sqrt{15}$.
4. $9+2(14)^{\frac{1}{2}}$.
5. $7+4(3)^{\frac{1}{2}}$.
6. $17-12\sqrt{2}$.
7. $2+(3)^{\frac{1}{2}}$.
8. $1+\frac{1}{2}\sqrt{3}$.
9. $2c-2(c^2-d^2)^{\frac{1}{2}}$.
10. $m+2\sqrt{mn-n^2}$.
11. $a-\sqrt{a^2-b^2}$.
12. $5x+x\sqrt{21}$.
13. $113-12(85)^{\frac{1}{2}}$.
14. $366+24\sqrt{210}$.
15. $540-14\sqrt{11}$.

195. Solution of Equations having the Unknown Numbers under Radical Signs.

1. Solve the equation $\sqrt{x^2-5}-x=-1$.

Transposing $-x$, $\qquad \sqrt{x^2-5}=x-1$.

Squaring both members, $\qquad x^2-5=x^2-2x+1$.

Transposing, $\qquad 2x=6$; whence, $x=3$.

(Substituting 3 for x in the given first member, and taking the positive value of the square root, the first member becomes

$$\sqrt{9-5}-3=2-3=-1;$$

which shows that the solution $x=3$ is correct.)

We then have the following rule:

Transpose the terms of the equation so that a surd term may stand alone in one member; then raise both members to a power of the same degree as the surd.

If surd terms still remain, repeat the operation.

The equation should be simplified as much as possible before performing the involution.

2. Solve the equation $\sqrt{2x-1}+\sqrt{2x+6}=7$.

Transposing $\sqrt{2x-1}$, $\sqrt{2x+6}=7-\sqrt{2x-1}$.

Squaring, $2x+6=49-14\sqrt{2x-1}+2x-1$.

Transposing, $14\sqrt{2x-1}=42$, or $\sqrt{2x-1}=3$.

Squaring, $2x-1=9$; whence, $x=5$.

3. Solve the equation $\sqrt{x-2}-\sqrt{x}=\dfrac{1}{\sqrt{x-2}}$.

Clearing of fractions, $x-2-\sqrt{x^2-2x}=1$.

Transposing, $-\sqrt{x^2-2x}=3-x$.

Squaring, $x^2-2x=9-6x+x^2$.

Transposing, $4x=9$, and $x=\dfrac{9}{4}$.

(If we put $x=\dfrac{9}{4}$, the given equation becomes

$$\sqrt{\frac{1}{4}}-\sqrt{\frac{9}{4}}=\frac{1}{\sqrt{\frac{1}{4}}}.) \qquad (1)$$

If we take the *positive* value of each square root, the above is not a true equation.

Authorities differ as to whether it is allowable in such instance to choose the *negative* value for one of the square roots. It seems more consistent to adhere to the signs expressed in the given equation. If this rule is followed, the above equation has no solution.

EXERCISE 40

Solve the following equations; verify each root:

1. $\sqrt{x+5}+2=5$.
2. $\sqrt{x+7}-\sqrt{x}=1$.
3. $\sqrt{x^2+4x-3}-\sqrt{x^2+x+6}=0$.
4. $\sqrt[3]{x^2+11}+4=7$.
5. $\sqrt{x^2-11}+1=x$.
6. $\sqrt{x-28}=14-\sqrt{x}$.
7. $\sqrt{x}+\sqrt{10-x}=\dfrac{12}{\sqrt{10-x}}$.
8. $\sqrt{x}+\sqrt{x+8}=\dfrac{12}{\sqrt{x+8}}$.
9. $\dfrac{\sqrt{x}+11}{\sqrt{x}-3}=\dfrac{\sqrt{x}+19}{\sqrt{x}-2}$.
10. $\dfrac{\sqrt{4x+5}+\sqrt{x+3}}{\sqrt{4x+5}-\sqrt{x+3}}=5$.

11. $\dfrac{3\sqrt{2x}+4}{\sqrt{2x}}=\dfrac{3\sqrt{2x}+2}{\sqrt{2x}+1}.$ 12. $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=\dfrac{1}{2}.$

13. $\sqrt{x+m}+\sqrt{x-n}=\sqrt{2m-n+3x}.$

14. $\sqrt{a-y}+\sqrt{b-y}=\sqrt{a-b}.$

15. $\sqrt{2s+3}-\sqrt{3s+3}=-\sqrt{s-10}.$

16. $\sqrt{1+x\sqrt{4-x}}=1+x.$ 19. $\sqrt[4]{x^2-5x-8}=\sqrt{x-4}.$

17. $x\sqrt{x-1}-\sqrt{x}=x.$ 20. $\dfrac{\sqrt{b^2+x}+\sqrt{c^2+x}}{\sqrt{b^2+x}-\sqrt{c^2+x}}=\dfrac{b}{c}.$

18. $\dfrac{\sqrt{b+x}+\sqrt{x}}{\sqrt{b+x}-\sqrt{x}}=b.$

VII. IMAGINARY NUMBERS

196. If a number involves an indicated even root of a negative number, it is called **imaginary**. Such numbers depend upon a new unit, $\sqrt{-1}$ or $(-1)^{\frac{1}{2}}$; as $\sqrt{-2}$, $\sqrt[4]{-3}$.

197. An imaginary number of the form $\sqrt{-a}$ is called a **pure imaginary** number, and the sum of a real and an imaginary is called a **complex** number; as $a+b\sqrt{-1}$.

198. Meaning of a Pure Imaginary Number.

If $\sqrt{a}$ is *real*, we define $\sqrt{a}$ as an expression such that, when raised to the second power, the result is a.

To find what meaning to attach to a pure imaginary number, we assume the above principle to hold when $\sqrt{a}$ is imaginary.

Thus, $\sqrt{-2}$ means an expression such that, when raised to the second power, the result is -2; that is, $(\sqrt{-2})^2$ or $(-2^{\frac{1}{2}})^2=-2$.

In like manner, $(\sqrt{-1})^2=(-1^{\frac{1}{2}})^2=-1$; etc.

OPERATIONS WITH IMAGINARY NUMBERS

199. By § 198, $(\sqrt{-5})^2 = (-5^{\frac{1}{2}})^2 = -5.$ (1)

Also, $$(\sqrt{5}\sqrt{-1})^2 = (\sqrt{5})^2(\sqrt{-1})^2 = 5(-1) = -5, \quad (2)$$

or $$(\sqrt{-5})^2 = (5^{\frac{1}{2}})^2 \cdot (-1^{\frac{1}{2}})^2 = 5(-1) = -5.$$

From (1) and (2), $(\sqrt{-5})^2 = (\sqrt{5}\sqrt{-1})^2.$

Whence, $$\sqrt{-5} = \sqrt{5}\sqrt{-1}, \text{ or } 5^{\frac{1}{2}}(-1)^{\frac{1}{2}}.$$

Then, *every imaginary square root can be expressed as the product of a real number by* $\sqrt{-1}$. *It is advisable to reduce every imaginary to this form before performing the indicated operations.*

$\sqrt{-1}$ is called the *imaginary unit*; it is often represented by i.

In all operations with imaginary numbers, it is advisable to reduce the number to the form $a + bi$ where a and b are real.

Ex. Add $\sqrt{-4}$ and $\sqrt{-36}$.

$$\sqrt{-4} = 2i, \quad \sqrt{-36} = 6i.$$

$$2i + 6i = 8i, \text{ or } 8\sqrt{-1}, \text{ or } 8(-1)^{\frac{1}{2}}.$$

The Powers of i: $$\sqrt{-1} = i^1;$$

$$(\sqrt{-1})^2 = -1 = i^2;$$

$$(\sqrt{-1})^3 = -\sqrt{-1} = i^3;$$

$$(\sqrt{-1})^4 = 1 = i^4;$$

$$(\sqrt{-1})^5 = \sqrt{-1} = i^5.$$

Note that the even powers of i are real, the odd powers imaginary, the fifth power like the first power, the sixth power like the second, etc.

Ex. 1. Multiply $\sqrt{-2}$ by $\sqrt{-3}$.

$$\sqrt{-2} = i\sqrt{2}, \quad \sqrt{-3} = i\sqrt{3}.$$

$$(i\sqrt{2})(i\sqrt{3}) = i^2\sqrt{6} = -\sqrt{6}, \text{ or } -(6)^{\frac{1}{2}}.$$

Ex. 2. Divide $(-40)^{\frac{1}{2}}$ by $(-5)^{\frac{1}{2}}$.

$$(-40)^{\frac{1}{2}} = i\,(40)^{\frac{1}{2}},\ (-5)^{\frac{1}{2}} = i\,(5)^{\frac{1}{2}}$$

$$\frac{i\,(40)^{\frac{1}{2}}}{i\,(5)^{\frac{1}{2}}} = (8)^{\frac{1}{2}} = 2\,(2)^{\frac{1}{2}}, \text{ or } 2\sqrt{2}.$$

EXERCISE 41

Simplify the following:

1. $\sqrt{-16} + \sqrt{-4}$.
2. $2\sqrt{-9} + 4\sqrt{-25} - 3\sqrt{-36}$.
3. $2\sqrt{-3} - 3\sqrt{-27} + 5\sqrt{-12}$.
4. $7\sqrt{-a^2} - 3\sqrt{-49\,a^2} - 2\sqrt{-4\,a^2}$.
5. $\frac{1}{3}\sqrt{-8} + \frac{1}{5}\sqrt{-32} - \frac{1}{6}\sqrt{-162}$.
6. Add $2 + \sqrt{-12}$ to $3 - 2\sqrt{-27}$.
7. From $8 - 6\sqrt{-121}$ subtract $5 + 2\sqrt{-169}$.
8. From $a - \sqrt{2\,b - b^2 - 1}$ take $b - \sqrt{2\,a - a^2 - 1}$.

Multiply the following:

9. $\sqrt{-2}$ by $\sqrt{-7}$.
10. $\sqrt{-4}$ by $\sqrt{-144}$.
11. $-\sqrt{-27}$ by $\sqrt{-6}$.
12. $-\sqrt{-432}$ by $-\sqrt{-75}$.
13. $\sqrt{-a^2}$, $\sqrt{-b^2}$, and $-\sqrt{-c^2}$.
14. $2 + \sqrt{-3}$ by $3 - 4\sqrt{-3}$.
15. $5 - 2\sqrt{-4}$ by $4 - 3\,i$.
16. $4\,i\sqrt{x} - 3\,i\sqrt{y}$ by $9\,i\sqrt{x} + \sqrt{-y}$.

Expand the following:

17. $(2 - \sqrt{-3})^2$.
18. $(3\sqrt{-2} + 2\sqrt{-3})^2$.
19. $(2\sqrt{-5} + 3\sqrt{-7})(2\sqrt{-5} - 3\sqrt{-7})$.
20. $(a - \sqrt{-b})^3$.

Divide the following:

21. $\sqrt{-18}$ by $\sqrt{-2}$.

22. $\sqrt{-54}$ by $-\sqrt{-3}$.

23. $-\sqrt{192}$ by $-\sqrt{-3}$.

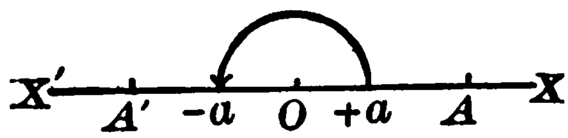
24. $-\sqrt{-96ab}$ by $\sqrt{-3ab}$.

25. $6i\sqrt{6}-\sqrt{384}$ by $-\sqrt{-6}$.

GRAPHICAL REPRESENTATION OF IMAGINARY NUMBERS

200. Let O be any point in the straight line XX'.

We may suppose any positive real number, $+a$, to be represented by the distance from O to A, a units to the right of O in OX.

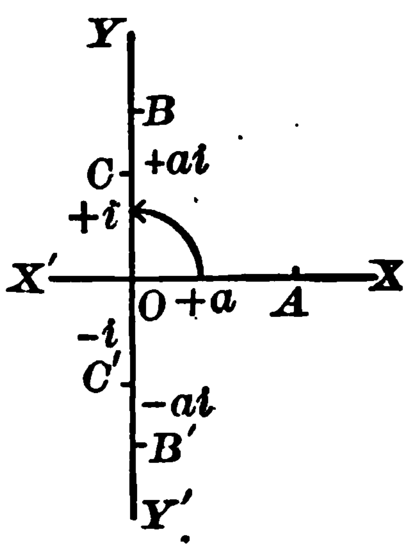


Then any negative real number, $-a$, may be represented by the distance from O to A', a units to the left of O in OX'.

201. Since $-a$ is the same as $(+a)\times(-1)$, it follows from § 200 that the product of $+a$ by -1 is represented by turning the line OA, which represents the number $+a$, through two right angles, in a direction opposite to the motion of the hands of a clock.

Then, in the product of any real number by -1, we may regard -1 as an operator which turns the line which represents the first factor through two right angles, in a direction opposite to the motion of the hands of a clock.

202. Graphical Representation of the Imaginary Unit i (§ 196).



By the definition of § 198, $-1 = i\times i$.

Then, since one multiplication by i, followed by another multiplication by i, turns the line which represents the first factor through *two* right angles, in a direction opposite to the hands of a clock, we may regard multiplication by i as turning the line through *one* right angle, in the same direction.

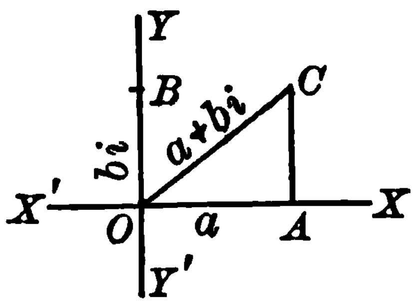
Thus, let XX' and YY' be straight lines intersecting at right angles at O.

Then, if $+a$ be represented by the line OA, where A is a units to the right of O in OX, $+ai$ may be represented by OB, and $-ai$ by OB', where B is a units above, and B' a units below, O, in YY'.

Also, $+i$ may be represented by OC, and $-i$ by OC', where C is one unit above, and C' one unit below, O, in YY'.

203. Graphical Representation of Complex Numbers.

We will now show how to represent the complex number $a+bi$.



Let XX' and YY' be straight lines intersecting at right angles at O.

Let a be represented by OA, to the right of O, if a is positive, to the left if a is negative.

Let bi be represented by OB, above O if b is positive, below if b is negative.

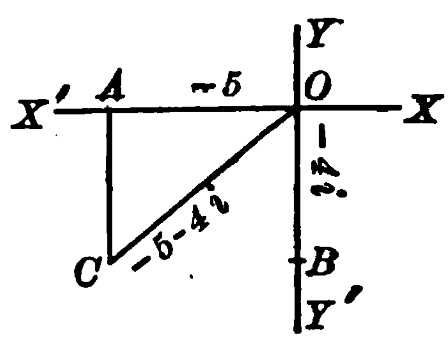
Draw line AC equal and parallel to OB, on the same side of XX' as OB, and line OC.

Then, OC is considered as representing the result of adding bi to a; that is, OC represents the complex number $a+bi$.

The figure represents the case in which both a and b are *positive.*

As another illustration, we will show how to represent the complex number $-5-4i$.

Lay off OA 5 units to the left of O in OX', and OB 4 units below O in YY'.



Draw line AC below XX', equal and parallel to OB, and line OC.

Then, OC represents $-5-4i$.

The complex number $a+bi$, if a is positive and b negative, will be represented by a line between OX and OY'; and if a is negative and b positive, by a line between OY and OX'.

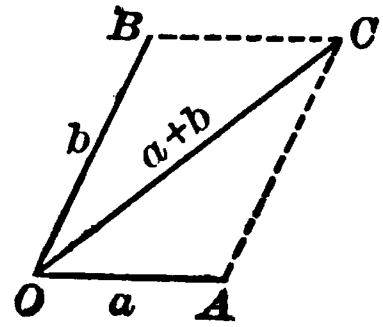
EXERCISE 42

Represent the following graphically:

1. $3i$.
2. $-6i$.
3. $4+i$.
4. $-1+2i$.
5. $2-5i$.
6. $-5-3i$.
7. $-7+4i$.

204. Graphical Representation of Addition.

We will now show how to represent the result of adding b to a, where a and b are any two real, pure imaginary, or complex numbers.



Let the line a be represented by OA, and the line b by OB.

Draw the line AC equal and parallel to OB, on the same side of OA as OB, and the line OC.

Then, OC is considered as representing the result of adding b to a; that is, OC represents $a+b$.

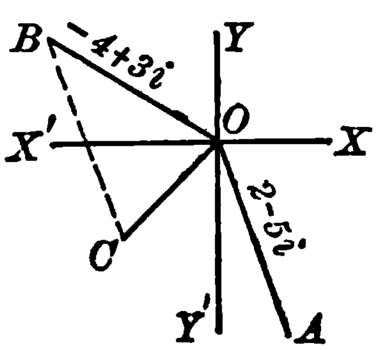
The method of § 203 is a special case of the above.

If a and b are both real, B will fall in OA, or in AO produced through O.

The same will be true if a and b are both pure imaginary.

If one of the numbers, a and b, is real, and the other pure imaginary, the lines OA and OB will be perpendicular.

As another illustration, we will show how to represent graphically the sum of the complex numbers $2-5i$ and $-4+3i$.



The complex number $2-5i$ is represented by the line OA, between OX and OY'.

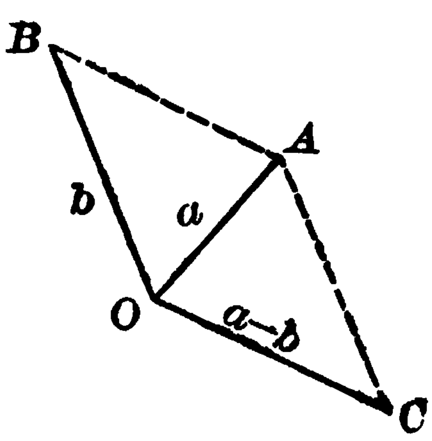
The complex number $-4+3i$ is represented by the line OB, between OY and OX'.

Draw the line BC equal and parallel to OA, on the same side of OB as OA, and the line OC.

Then, the line OC represents the result of adding $-4+3i$ to $2-5i$.

205. Graphical Representation of Subtraction.

Let a and b be any two real, pure imaginary, or complex numbers.



Let a be represented by OA, and b by OB; and complete the parallelogram $OBAC$.

By § 204, OA represents the result of adding the number represented by OB to the number represented by OC.

That is, if b be added to the number represented by OC, the sum is equal to a; hence, $a-b$ is represented by the line OC.

EXERCISE 43

Represent the following graphically:

1. The sum of $4i$ and $3-5i$.
2. The sum of $-5i$ and $-1+6i$.
3. The sum of $2+4i$ and $5-3i$.
4. The sum of $-6+2i$ and $-4-7i$.

5. Represent graphically the result of *subtracting* the second expression from the first, in each of the above examples.

VIII. QUADRATIC EQUATIONS

206. A **quadratic equation** is an equation in which the highest power of the unknown number is the second.

207. The first power of the unknown number may or may not appear. If the equation does not contain the first degree of the unknown, the roots are of the same absolute value but of different sign. *E.g.* $x^2 = a^2$; then, $(x+a)(x-a)=0$, or $x = a$, $x = -a$.

The equation may also be solved by extracting the square root of each member of the equation, whence, $x = \pm a$.

208. If the equation contains both the first and second powers of the unknown, the first member must be reduced to the form $a^2 + 2ab + b^2$ before extracting the square root. Such transformation of the equation is called completing the square.

209. A quadratic equation containing the first and second powers of the unknown number is called an **affected quadratic.** An equation containing the second degree only of the unknown number is a **pure quadratic.**

AFFECTED QUADRATIC EQUATIONS

210. First Method of Completing the Square.

By transposing the terms involving x to the first member, and all other terms to the second, and then dividing both

members by the coefficient of x^2, any affected quadratic equation can be reduced to the form $x^2 + px = q$.

We then add to both members such an expression as will make the first member a trinomial perfect square (§ 103, II); an operation which is termed *completing the square.*

Ex. Solve the equation $x^2 + 3x = 4$.

A trinomial is a perfect square when its first and third terms are perfect squares and positive, and its second term plus or minus twice the product of their square roots (§ 103, II).

Then, the square root of the third term is equal to the second term divided by twice the square root of the first.

Hence, the *square root* of the expression which must be added to $x^2 + 3x$ to make it a perfect square is $3x \div 2x$, or $\frac{3}{2}$.

Adding to both members the square of $\frac{3}{2}$, we have

$$x^2 + 3x + (\tfrac{3}{2})^2 = 4 + \tfrac{9}{4} = \tfrac{25}{4}.$$

Equating the square root of the first member to the $\pm$ square root of the second, we have

$$x + \tfrac{3}{2} = \pm \tfrac{5}{2}.$$

Transposing $\frac{3}{2}$, $\qquad x = -\frac{3}{2} + \frac{5}{2}$ or $-\frac{3}{2} - \frac{5}{2} = 1$ or -4.

We then have the following rule:

Reduce the equation to the form $x^2 + px = q$.

Complete the square, by adding to both members the square of one-half the coefficient of x.

Equate the square root of the first member to the $\pm$ square root of the second, and solve the linear equations thus formed.

211. The objection to the method of § 210 is that in dividing by the coefficient of x^2, or in adding the square of one-half the coefficient of x, fractions which make the solution cumbersome may be introduced.

212. If the coefficient of x^2 is a perfect square, it is sometimes convenient to complete the square directly by the principle stated in § 210; that is, *by adding to both members the square of the quotient obtained by dividing the coefficient of x by twice the square root of the coefficient of x^2.*

Ex. Solve the equation $9x^2 - 5x = 4$.

Adding to both members the square of $\frac{5}{2 \times 3}$, or $\frac{5}{6}$,

$$9x^2 - 5x + (\tfrac{5}{6})^2 = 4 + \tfrac{25}{36} = \tfrac{169}{36}.$$

Extracting square roots, $3x - \frac{5}{6} = \pm \frac{13}{6}$.

Then, $3x = \frac{5}{6} \pm \frac{13}{6} = 3$ or $-\frac{4}{3}$, and $x = 1$ or $-\frac{4}{9}$.

213. Second Method of Completing the Square.

Every affected quadratic equation can be reduced to the form $ax^2 + bx + c = 0$, or $ax^2 + bx = -c$.

Multiplying both members by $4a$, we have

$$4a^2x^2 + 4abx = -4ac.$$

We complete the square by adding to both members the square of

$$\frac{4ab}{2 \times 2a} \text{ (§ 212), or } b.$$

Then, $4a^2x^2 + 4abx + b^2 = b^2 - 4ac$.

Extracting square roots, $2ax + b = \pm\sqrt{b^2 - 4ac}$.

Transposing, $2ax = -b \pm \sqrt{b^2 - 4ac}$.

Whence,
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \quad (1)$$

We then have the following rule:

Reduce the equation to the form $ax^2 + bx = -c$.
Multiply both members by four times the coefficient of x^2, and add to each the square of the coefficient of x in the given equation.

The advantage of this method over the preceding is in avoiding fractions in completing the square.

This method is sometimes called the Hindoo Method.

The result of the solution of $ax^2 + bx + c = 0$ may be used as a formula for solving any quadratic equation. Before applying the formula the equation must be reduced to the form

$$ax^2 + bx + c = 0.$$

Ex. Solve $2x^2 - 7x = -3$.

$$2x^2 - 7x + 3 = 0.$$

Here $a = 2,\ b = -7,\ c = 3$; substituting in (1),

$$x = \frac{7 \pm \sqrt{7^2 - 4 \cdot 2 \cdot 3}}{2 \cdot 2} = \frac{7 \pm 5}{4} = 3 \text{ or } \frac{1}{2}.$$

EXERCISE 44

Solve by the first method: (Verify each result.)

1. $x^2 - 12x + 32 = 0.$
2. $z^2 + 7z - 30 = 0.$
3. $4y^2 - 7y = -3.$
4. $16x^2 - 8x - 35 = 0.$
5. $3m^2 - 26 = 9m^2 - 80.$
6. $t^2 + t - 30 = 0.$
7. $6z^2 + 4 = -11z.$
8. $4x^2 - 3x = 7.$
9. $\frac{z^2}{3} - \frac{z}{2} - \frac{35}{6} = 0.$
10. $\frac{3x^2}{x^2 - 7x + 6} + \frac{2x - 6}{x - 6} = 1 - \frac{2}{x - 1}.$

Solve by second method: (Verify each result.)

11. $(3k + 2)(2k + 3) = (k - 3)(2k - 4).$
12. $\frac{30}{x} - \frac{30}{x + 1} = 1.$
13. $\sqrt{m + 2} + \sqrt{3m + 4} = 8.$
14. $(y - 3)^3 - (y + 2)^3 = -65.$
15. $\sqrt{5 + x} + \sqrt{5 - x} = \frac{12}{\sqrt{5 - x}}.$
16. $\frac{d + 3}{d - 2} - \frac{d + 4}{d} = \frac{3}{2}.$

Note 1: In solving equations involving fractions or radicals reject any root which does not satisfy the *given* equation.

Solve by means of the formula in § 213, (1): (Verify each result.)

17. $3x^2 - 2x = 40.$

18. $9x^2 + 18x = -8.$

19. $\frac{5}{6z} - \frac{13}{9z^2} = \frac{1}{18}.$

20. $\frac{1}{x+3} - \frac{1}{x-5} = \frac{x^2-17}{x^2-2x-15}.$

21. $\frac{y-c}{y+c} - \frac{y+c}{y-c} = \frac{y^2-5c^2}{y^2-c^2}.$

22. $\frac{1}{z-2} + \frac{7z}{24(z+2)} = \frac{15}{z^2-4}.$

23. $\frac{1}{x+a} + \frac{1}{a} + \frac{1}{x+b} + \frac{1}{b} = 0.$

24. $S = Vt + \frac{1}{2}gt.^2$ Solve for t.

25. $\frac{x-2}{x-4} - \frac{x+2}{x+3} - \frac{x-2}{x-6} = -1.$ (See Note 2.)

26. $\frac{x+1}{x-1} + \frac{x+2}{x-2} + \frac{x+3}{x-3} = 3.$

Note 2: In solving fractional equations containing improper fractions the operations are greatly simplified by reducing the fractions to mixed numbers and then combining the integers thus obtained.

Note 3: An equation is said to be in the *quadratic form* when it is expressed in three terms, two of which contain the unknown number, and *the exponent of the unknown number in one of these terms is twice its exponent in the other;* as,

$$x^6 - 6x^3 = 16; \; x^3 + x^{\frac{3}{2}} - 72 = 0; \text{ etc.}$$

Equations in the quadratic form may be readily solved by the rules for quadratics.

1. Solve the equation $x^6 - 6x^3 = 16$.

Completing the square by the rule of § 210,

$$x^6 - 6x^3 + 9 = 16 + 9 = 25.$$

Extracting square roots, $x^3 - 3 = \pm 5.$

Then, $x^3 = 3 \pm 5 = 8 \text{ or } -2.$

Extracting cube roots, $x = 2 \text{ or } -\sqrt[3]{2}.$

There are also four imaginary roots, which may be found by the method of §§ 110; 213, (1).

Solve the equation $2x + 3\sqrt{x} = 27.$

Since $\sqrt{x}$ is the same as $x^{\frac{1}{2}}$, this is in the quadratic form.

Multiplying by 8, and adding 3^2 to both members § (213),

$$16x + 24\sqrt{x} + 9 = 216 + 9 = 225.$$

Extracting square roots, $4\sqrt{x} + 3 = \pm 15.$

Then, $4\sqrt{x} = -3 \pm 15 = 12 \text{ or } -18.$

Whence, $\sqrt{x} = 3 \text{ or } -\frac{9}{2}$, and $x = 9 \text{ or } \frac{81}{4}.$

EXERCISE 45

Solve the following equations and verify each root:

1. $3x^2 - 4x = 4.$
2. $7x^2 - 17x = 2x^2 + 22.$
3. $4y^2 + 9y - 13 = 0.$
4. $2t = 10 - t^2 + 5t.$
5. $6v^2 - 14v = 9v - 22.$
6. $\frac{x-3}{x+3} + \frac{x+5}{x-11} = -\frac{8}{7}.$
7. $\frac{3m-7}{m(m+2)} - \frac{1}{3(m-2)} = \frac{2-m}{m^2-4}.$
8. $\frac{4}{9-q} + \frac{15}{10-q} = 4.$
9. $\frac{x+1}{x+2} + \frac{x+2}{x+3} = 3\frac{1}{2}.$
10. $1 + 2\sqrt{3x+2} = \frac{21}{\sqrt{3x+2}}.$
11. $\sqrt{x-b} + 2\sqrt{2b} = \frac{6b}{\sqrt{x-b}}.$
12. $\frac{11}{3v-4a} - \frac{20}{2v+a} - \frac{3a}{6v^2-5av-4a^2} = \frac{6}{5a}.$

13. $\dfrac{a}{z-a}+\dfrac{b}{b-z}=\dfrac{a^2-b^2}{ab}.$

14. $\dfrac{x+a+c}{2}+\dfrac{3\,ac+6\,bc-9\,c^2}{x}=a+b+2\,c.$

15. $x^2-2\,ax+6\,b^2=3\,a^2+7\,ab-5\,bx.$

16. $(2\,a-b+5\,c)x^2+(b-a+4\,c)x+2\,b-3\,a-c=0$

17. $\dfrac{1}{d+a}+\dfrac{1}{a}+\dfrac{1}{d+b}+\dfrac{1}{b}=0.$ Solve for d.

18. $\dfrac{\sqrt{m^2+x^2}+\sqrt{m^2-x^2}}{\sqrt{m^2+x^2}-\sqrt{m^2-x^2}}=\dfrac{m}{x}.$

19. $x^4-7\,x^2+12=0.$

20. $x^6-7\,x^3=8.$

21. $6\,x^{-2}-11\,x^{-1}=35.$

22. $x^{\frac{1}{2}}-x^{\frac{1}{4}}-6=0.$

23. $x^3-35\,x^{\frac{3}{2}}=-216.$

24. $x^2+2\,x+10+\sqrt{x^2+2\,x+10}=30.$

25. $x^2+3\,ax-53\,a^2=2\,ax+3\,a.$

26. $x^{-\frac{4}{3}}-29\,x^{-\frac{2}{3}}=-100.$

27. $x^2+14\sqrt{x^2+7\,x-26}=58-7\,x.$

28. $5\,(x+2)^{\frac{1}{2}}+(x+2)=36.$

29. $(x^2+4\,x+2)^2=31+2\,(x^2+4\,x+2).$

30. $x^4-8\,x^3+10\,x^2+24\,x-315=0.$

31. What number is that to which if you add its square the sum will be 42?

32. A rectangular field is 40 rods longer than it is wide. By doubling its length and decreasing its width by 15 rods, the area is unchanged. Find dimensions of the field.

33. The difference between two numbers is 7, and the difference between their cubes is 1267. Find the numbers.

34. The denominator of a fraction is 3 more than its numerator and by adding the fraction to its reciprocal the sum is $2\frac{9}{28}$. What is the fraction?

35. There is a number consisting of two digits whose sum is 11. If from the number 3 times the product of the digits is subtracted, the remainder will equal the sum of the digits. Find the number.

36. A man sells goods for \$120, gaining a per cent equal to $\frac{1}{5}$ the cost of the goods. What was the cost of the goods?

37. A picture 13 inches by 8 inches is surrounded by a frame of uniform width whose area is 162 square inches. Find the width of the frame.

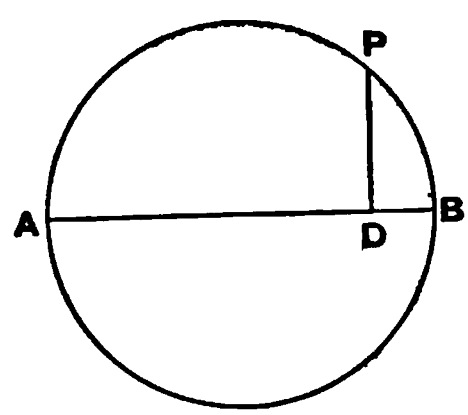
38. A man put \$2400 in a savings bank which paid interest semiannually. At the end of a year he found that he had to his credit \$2496.96. What interest did the bank pay?

39. A number of people plan an excursion which is to cost them \$30. It is found later that 3 of the party cannot go, which increases the cost 50 cents to each member. How many are there in the party and what did each one pay?

40. A and B start together for a 6-mile walk. A's rate per hour is $\frac{1}{2}$ mile more than B's, and he finds he can reach his destination in 24 minutes less time than B. What is the rate of each?

41. An open rectangular box is 8 inches high. Its length is 4 more than its width. Its volume is 768 cubic inches. Find its inside dimensions.

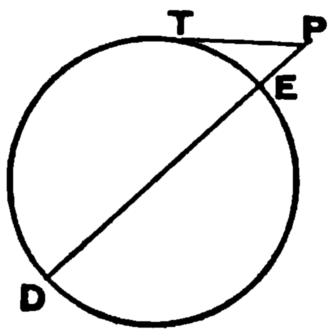
42. In a given circle APB, a perpendicular DP, dropped from a point P in the circumference to the diameter AB, is a mean proportional between the segments, AD and DB, of the diameter. If the radius of the circle is 12 and DP is $2\sqrt{5}$, how far is D from B?



43. An open rectangular box 5 inches deep (inside measure) is made of 1-inch lumber. Its length is 1 inch less than twice its width. The difference between the volumes when inside and outside measurements are taken is 271 cubic inches. How much sheet metal will be needed for lining the sides and bottom of the box?

44. Two lines AB and CD intersect at O in such a manner that $AO \cdot OB = CO \cdot OD$. If $CD = 14$, $AO = 15$, and $AB = 18$, find CO.

45. A has a lease on a square room. He sublet to B a part 10 feet wide along one entire side of the room, at a rental of $160 per month. The part of the room retained by A contained 704 square feet. How much rental per square foot did B pay? Explain your negative roots.



46. A tangent, PT, to a circle is a mean proportional between the whole secant PD and the external segment PE. If PT is 12, the radius 5, and PD passes through the center, find PE.

47. The upper base and the altitude of a trapezoid are equal, the lower base is 20 and the area is 112. Find the upper base.

48. The length of a rectangle is $\sqrt{2}$ more than the side of a given square, and its breadth is $\sqrt{2}$ less than a side of the same square. The area of the rectangle is 1. Find the dimensions of the rectangle correct to three decimal places.

THEORY OF QUADRATIC EQUATIONS

214. Number of Roots.

A quadratic equation cannot have more than two different roots.

Every quadratic equation can be reduced to the form

$$ax^2 + bx + c = 0.$$

If possible, let this have three different roots, r_1, r_2, and r_3.

Then, $$ar_1^2 + br_1 + c = 0, \quad (1)$$

$$ar_2^2 + br_2 + c = 0, \quad (2)$$

and $$ar_3^2 + br_3 + c = 0. \quad (3)$$

Subtracting (2) from (1), $a(r_1^2 - r_2^2) + b(r_1 - r_2) = 0.$

Then, $a(r_1 + r_2)(r_1 - r_2) + b(r_1 - r_2) = 0,$

or, $(r_1 - r_2)(ar_1 + ar_2 + b) = 0.$

Then, by § 110, either $r_1 - r_2 = 0$, or $ar_1 + ar_2 + b = 0$.

But $r_1 - r_2$ cannot equal 0, for, by hypothesis, r_1 and r_2 are different.

Whence, $$ar_1 + ar_2 + b = 0. \tag{4}$$

In like manner, by subtracting (3) from (1), we have

$$ar_1 + ar_3 + b = 0. \tag{5}$$

Subtracting (5) from (4), $ar_2 - ar_3 = 0$, or $r_2 - r_3 = 0$.

But this is impossible, for, by hypothesis, r_2 and r_3 are different; hence, a quadratic equation cannot have more than two different roots.

215. The graphs of quadratic equations can be readily constructed by the method used in §§ 44–48.

Construct the graph of $$x^2 - x - 6 = 0. \tag{1}$$

Placing the first member of the equation equal to y, we have

$$x^2 - x - 6 = y. \tag{2}$$

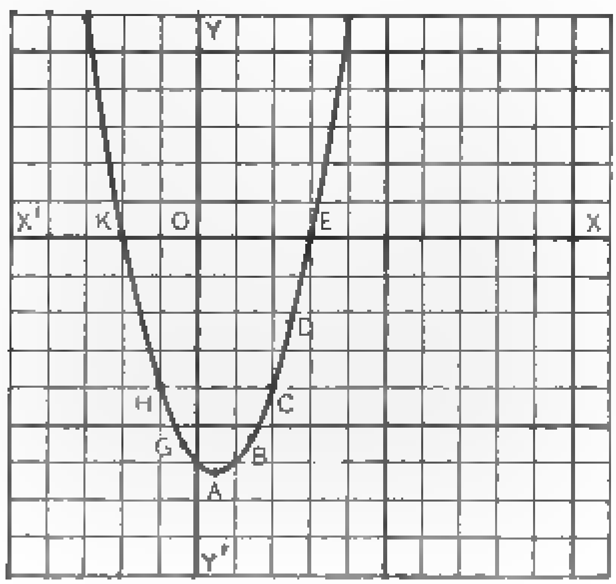
Assigning values to x, we obtain corresponding values of y. For example,

Substituting $x = 0$ in (2), we have $y = -6$,

Substituting $x = 2$ in (2), we have $y = -4$, etc.

$x^2 - x - 6 = y$

x	y
0	-6
$\frac{1}{2}$	$-6\frac{1}{4}$ (*A*)
1	-6
$\frac{3}{2}$	$-5\frac{1}{4}$ (*B*)
2	-4 (*C*)
$\frac{5}{2}$	$-2\frac{1}{4}$ (*D*)
3	0 (*E*)
4	6
$-\frac{1}{2}$	$-5\frac{1}{4}$ (*G*)
-1	-4 (*H*)
-2	0 (*K*)
-3	6



Solving $$x^2 - x - 6 = 0,$$

or $$(x-3)(x+2) = 0,$$

we have, $$x = 3 \text{ or } -2.$$

These values, $x = 3$, $x = -2$, are the abscissas of the points where the curve crosses the x-axis, the curve showing in a graphical way why a quadratic equation has two roots.

The graph of every equation of the form $x^2 + px - q = 0$ or $ax^2 + bx + c = 0$ is a curve of the above form and is called a *parabola.*

216. Sum of Roots and Product of Roots.

Let r_1 and r_2 denote the roots of $ax^2 + bx + c = 0$.

By § 213, (1), $r_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$, and $r_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.

Adding these values, $$r_1 + r_2 = \frac{-2b}{2a} = -\frac{b}{a}.$$

Multiplying them together,

$$r_1 r_2 = \frac{b^2 - (b^2 - 4ac)}{4a^2} \text{ (§ 103, I)} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

Hence, if a quadratic equation is in the form $ax^2 + bx + c = 0$, the sum of the roots equals minus the coefficient of x divided by the coefficient of x^2, and the product of the roots equals the independent term divided by the coefficient of x^2.

217. Formation of Quadratic Equations.

By aid of the principles of § 216, a quadratic equation may be formed which shall have any required roots.

For, let r_1 and r_2 denote the roots of the equation

$$ax^2 + bx + c = 0, \text{ or } x^2 + \frac{bx}{a} + \frac{c}{a} = 0. \quad (1)$$

Then by § 216, $$\frac{b}{a} = -r_1 - r_2, \text{ and } \frac{c}{a} = r_1 r_2.$$

Substituting these values in (1), we have

$$x^2 - r_1 x - r_2 x + r_1 r_2 = 0.$$

Or, $$(x - r_1)(x - r_2) = 0.$$

Therefore, to form a quadratic equation which shall have any required roots,

Subtract each of the roots from x, and place the product of the resulting expressions equal to zero.

Ex. Form the quadratic whose roots shall be 4 and $-\frac{7}{4}$.

By the rule, $$(x - 4)(x + \tfrac{7}{4}) = 0.$$

Multiplying by 4, $$(x - 4)(4x + 7) = 0; \text{ or, } 4x^2 - 9x - 28 = 0.$$

DISCUSSION OF GENERAL EQUATION

218. The roots of a quadratic equation may take several forms:

1. The roots may be rational, unequal, of the same sign.
2. The roots may be rational, unequal, of opposite sign.
3. The roots may be rational, equal.
4. The roots may be irrational, unequal.
5. The roots may be irrational, equal.
6. The roots may be irrational and the number under the radical sign *negative.*

These forms and the causes for their existence are at once seen when one considers the formula in § 213.

By § 213, the roots of $ax^2 + bx + c = 0$ are

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

We will now discuss these results for all possible real values of a, b, and c.

I. $b^2 - 4ac$ *positive.*

In this case, r_1 and r_2 are *real* and *unequal.*

II. $b^2 - 4ac = 0$.

In this case, r_1 and r_2 are *real* and *equal.*

III. $c = 0$.

In this case, the equation takes the form

$$ax^2 + bx = 0; \text{ whence } x = 0 \text{ or } -\frac{b}{a}.$$

Hence, the roots are both *real,* one being *zero.*

IV. $b = 0$, *and* $c = 0$.

In this case, the equation takes the form $ax^2 = 0$.

Hence, both roots equal zero.

V. $b^2 - 4ac$ *negative.*

In this case, r_1 and r_2 are *imaginary* (§ 196).

VI. $b = 0$.

In this case, the equation takes the form

$$ax^2 + c = 0; \text{ whence, } x = \pm\sqrt{-\frac{c}{a}}.$$

If a and c are of unlike sign, the roots are *real, equal in absolute value,* and *unlike in sign.*

If a and c are of like sign, both roots are *imaginary.*

The roots are both *rational,* or both *irrational,* according as $b^2 - 4ac$ is, or is not, a perfect square.

219. It is evident that irrational roots, whether real or imaginary, must occur in *conjugate* pairs.

That is, in an equation of the form of $ax^2 + bx + c = 0$, where a, b, c are real, if one root is of the form $k + \sqrt{h}$ the other must be $k - \sqrt{h}$ where k and h are real.

EXERCISE 46

Find by inspection the sum and product of the roots of the following:

1. $x^2 - 2x - 35 = 0.$
2. $x^2 + 15x + 36 = 0.$
3. $2x^2 + 7x - 4 = 0.$
4. $5x^2 - 13x = -6.$
5. $x^2 + ax - bx = ab.$
6. $cdx^2 + d^2x = c^2x + cd.$
7. $x^2 - 2\sqrt{2}x - 2 = 0.$
8. One root of $8x^2 - 2x - 15 = 0$ is $-1\frac{1}{4}$; find the other.
9. One root of $6x^2 + 11x - 2 = 0$ is $\frac{1}{6}$; find the other.
10. One root of $2x^3 - 8x^2 + 2x + 12 = 0$ is 2; find the others.
11. One root of $m^3 - 7m + 6 = 0$ is -3; find the others.
12. If r_1 and r_2 are the roots of $x^2 + x + 1 = 0$, what does $r_1^2 + r_2^2$ equal? $r_1^3 + r_2^3$?

Form the equations whose roots are:

13. 2, 3.
14. $-1, 4.$
15. $\frac{1}{2}, -3.$
16. $-\frac{3}{5}, -\frac{2}{7}.$
17. $0, -\frac{5}{6}.$
18. $a, 6a.$
19. $a + \sqrt{b}, a - \sqrt{b}.$
20. $2 + \sqrt{-3}, 2 - \sqrt{-3}.$
21. $3c - d, -2c + 5d.$

22. $\dfrac{\sqrt{2k}-5\sqrt{g}}{2}$, $\dfrac{\sqrt{2k}+5\sqrt{g}}{2}$.

23. $6, -\frac{1}{4}, 0.$

Determine by inspection the nature of the roots of the following:

24. $x^2+7x+12=0.$

25. $x^2+8x=-16.$

26. $x^2+2x-1=0.$

27. $x^2+2x+3=0.$

28. $2x^2+7x=3.$

29. $4x^2-16=0.$

30. $2x^2=15x+18.$

31. $x^2-x=12.$

32. $10x^2-x=2.$

33. $23x-6=7x^2.$

34. $16x^2+24x+9=0.$

35. $5x^2+3x=-2.$

GRAPHS

220. The nature of the roots discussed in § 218 is illustrated by the use of graphs:

Fig. 1

x	y
0	-8
1	-9
2	-8
3	-5
4	0
5	7
-1	-5
-2	0
-3	7

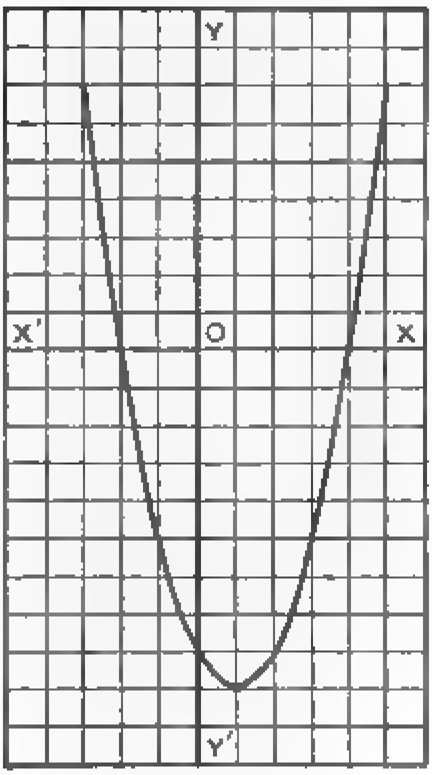


Fig. 1. $x^2-2x-8=0$
$b^2-4ac>0$

Fig. 2

x	y
0	1
1	0
2	1
3	4
4	9
5	16
-2	9
-3	16

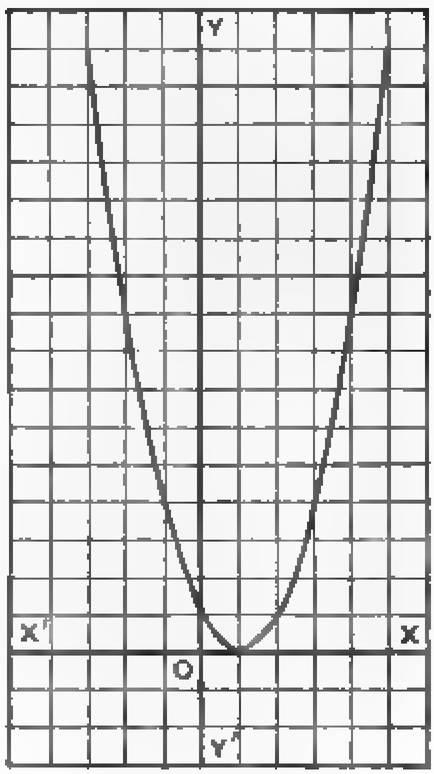
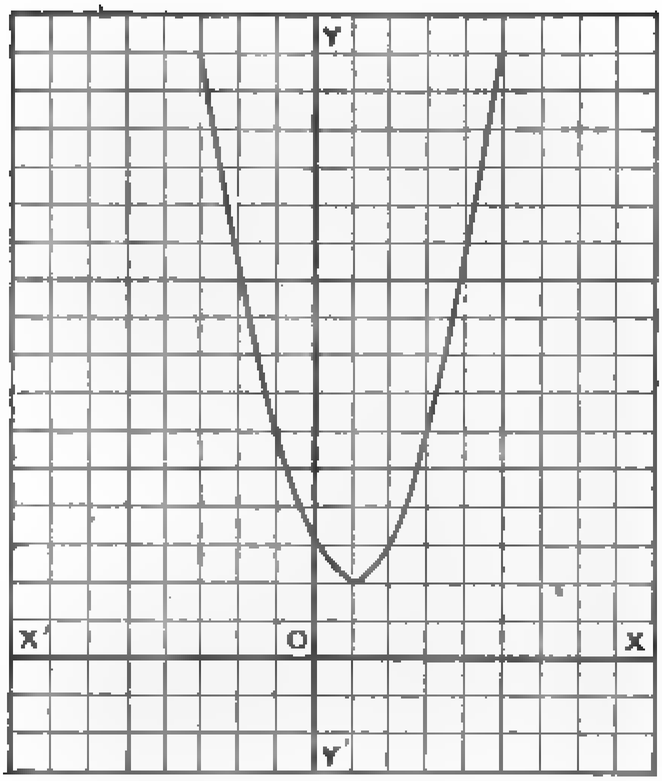


Fig. 2. $x^2-2x+1=0$
$b^2-4ac=0$

Fig. 3

x	y
0	3
1	2
2	3
3	6
4	11
-1	6
-2	11
-3	18



3. $x^2 - 2x + 3 = 0$

$b^2 - 4ac < 0$

In Fig. 1, the curve crosses the x-axis at points whose abscissas are 4, -2; the abscissas of these points being the values of x found in solving the equation. In Fig. 2, the intersection points coincide and we have two values of x each equal to 1. In Fig. 3, the curve and the x-axis do not coincide.

EXERCISE 47

Plot the curves:

1. $f(x) = x^2 + 6x + 8.$ (§§ 47, 220.)
2. $f(x) = x^2 - 6x + 8.$
3. $f(x) = x^2 - 9.$
4. $f(x) = x^2 - 6x + 9.$
5. $f(x) = x^2 + 2x + 4.$

221. Many problems in Physics are dependent on the laws of proportion and variation. The solution of such problems is often obtained more readily by graphical means than by algebraic solution.

Ex. 1. *Graphical representation of a direct proportion.*

When a man is running at a constant speed, the distance which he travels in a given time is directly proportional to his speed. The algebraic expression of this relation is $\frac{d_1}{d_2}=\frac{s_1}{s_2}$, or $d=ms$. (See § 161.)

Now, if we plot successive values of the distance, d, which correspond to various speeds, s, in precisely the same manner in which we plotted successive values of x and y in §§ 44–48, we obtain as the graphical picture of the relation between s and d a straight line passing through the origin. (See Fig. 1.)

This is the graph of any direct proportion.

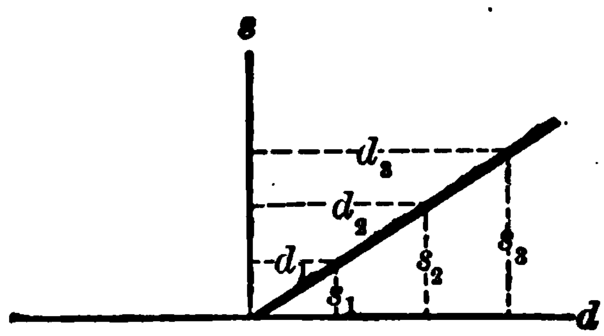


FIG. 1.

Ex. 2. *Graphical representation of an inverse proportion.*

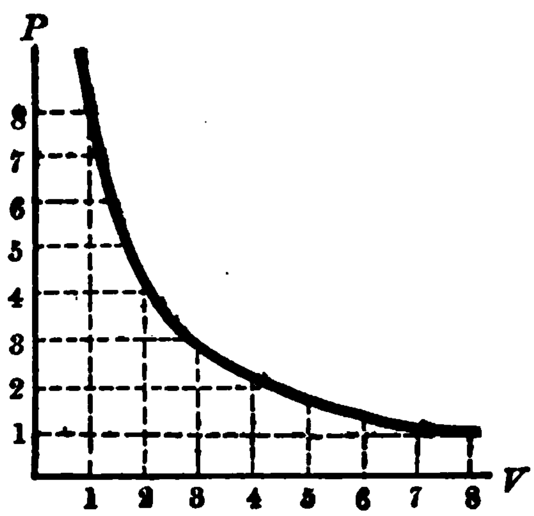


FIG. 2.

The volume which a given body of gas occupies when the pressure to which it is subjected varies has been found to be inversely proportional to the pressure under which the gas stands; we have seen that the algebraic statement of this relation is $\frac{V_1}{V_2}=\frac{P_2}{P_1}$.

$$\frac{V_1}{V_2}=\frac{P_2}{P_1}, \text{ or } V=\frac{m}{P}.$$

$V=1,\quad P=m.\quad V=5,\quad P=\frac{m}{5}.$

$V=2,\quad P=\frac{m}{2}.\quad V=6,\quad P=\frac{m}{6}.$

$V=3,\quad P=\frac{m}{3}.\quad V=7,\quad P=\frac{m}{7}.$

$V=4,\quad P=\frac{m}{4}.\quad V=8,\quad P=\frac{m}{8}.$

If we plot successive values of V and P in the manner indicated in §§ 44–48, we obtain a graph of the form shown in Fig. 2.

This is the *graphical representation of any inverse proportion*; the curve is called an **equilateral hyperbola.**

Ex. 3. *The path traversed by a falling body projected horizontally.*

When a body is thrown horizontally from the top of a tower, if it were not for gravity, it would move on in a horizontal direction indefinitely, traversing exactly the same distance in each succeeding second.

Hence, if V represents the velocity of projection, the horizontal distance, H, which it would traverse in any number of seconds, t, would be given by the equation $H = Vt$.

On account of gravity, however, the body is pulled downward, and traverses in this direction in any number of seconds a distance which is given by the equation $S = \frac{1}{2}gt^2$.

To find the actual path taken by the body, we have only to plot successive values of H and S, in the manner in which we plotted the successive values of x and y, in §§ 44–48.

Thus, at the end of 1 second the vertical distance S_1 is given by $S_1 = \frac{1}{2}g \times 1^2 = \frac{1}{2}g$; at the end of 2 seconds we have, $S_2 = \frac{1}{2}g \times 2^2 = \frac{4}{2}g$; at the end of 3 seconds, $S_3 = \frac{1}{2}g \times 3^2 = \frac{9}{2}g$; at the end of 4 seconds, $S_4 = \frac{1}{2}g \times 4^2 = \frac{16}{2}g$; etc.

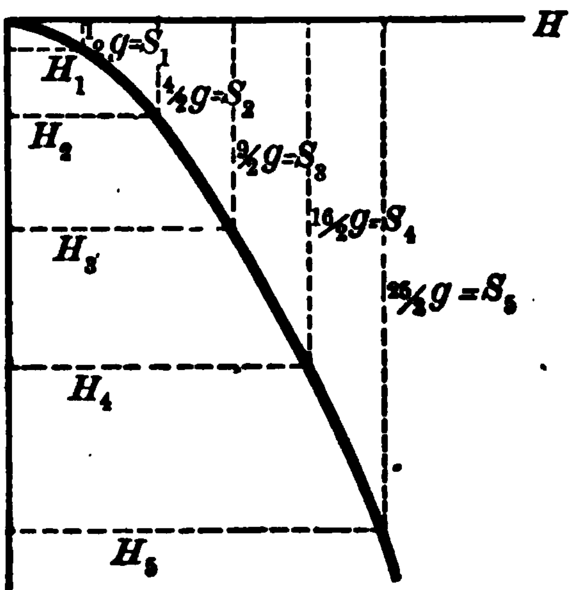


FIG. 3.

On the other hand, at the end of 1 second we have $H_1 = V$; at the end of 2 seconds, $H_2 = 2V$; at the end of 3 seconds, $H_3 = 3V$; at the end of 4 seconds, $H_4 = 4V$.

If, now, we plot these successive values of H and S, we obtain the graph shown in Fig. 3.

This is the path of the body; it is a parabola. (§ 226, Ex. 2.)

Ex. 4. *Graph of relation between the temperature and pressure existing within an air-tight boiler containing only water and water vapor.*

One use of graphs in physics is to express a relation which is found by experiment to exist between two quantities, which cannot be represented by any simple algebraic equation.

For example, when the temperature of an air-tight boiler which contains only water and water vapor is raised, the pressure within the boiler increases also; thus we find by direct experiment that when the temperature of the boiler is 0° centigrade, the pressure which the vapor exerts will support a column of mercury 4.6 millimeters high.

When the temperature is raised to 10°, the mercury column rises to 9.1 millimeters; at 30° the column is 31.5 millimeters long, etc.

To obtain a simple and compact picture of the relation between temperature and pressure, we plot a succession of temperatures, *e.g.* 0°, 10°, 20°, 30°, 40°, 50°, 60°, 70°, 80°, 90°, 100°, in the manner in which we plotted successive values of x in §§ 44–48, and then plot the corresponding values of pressure obtained by experiment in the manner in which we plotted the y's in §§ 44–48; we obtain the graph shown in Fig. 4.

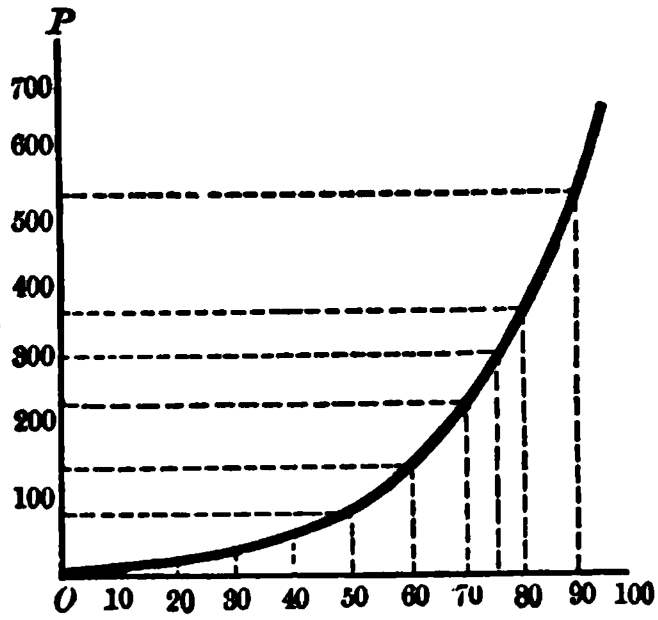


FIG. 4.

From this graph we can find at once the pressure which will exist within the boiler at any temperature.

For example, if we wish to know the pressure at 75° centigrade, we observe where the vertical line which passes through 75° cuts the curve and then run a horizontal line from this point to the point of intersection with the line OP.

This point is found to be at 288; hence the pressure within the boiler at 75° centigrade is 288 millimeters.

EXERCISE 48

PROBLEMS IN PHYSICS

1. When the force which stretches a spring, a straight wire, or any elastic body is varied, it is found that the displacement produced in the body is always directly proportional to the force which acts upon it; *i.e.* if d_1 and d_2 represent any two displacements, and f_1 and f_2 respectively the forces which produce them, then the algebraic statement of the above law is

$$\frac{d_1}{d_2}=\frac{f_1}{f_2}. \tag{1}$$

If a force of 2 pounds stretches a given wire .01 inch, how much will a force of 20 pounds stretch the same wire?

2. If the same force is applied to two wires of the same length and material, but of different diameters, D_1 and D_2, then

the displacements d_1 and d_2 are found to be inversely proportional to the squares of the diameters, *i.e.*

$$\frac{d_1}{d_2} = \frac{D_2^2}{D_1^2}. \qquad (2)$$

If a weight of 100 kilograms stretches a wire .5 millimeter in diameter through 1 millimeter, how much elongation will the same weight produce in a wire 1.5 millimeters in diameter?

3. If the same force is applied to two wires of the same diameter and material, but of different lengths, l_1 and l_2, then it is found that

$$\frac{d_1}{d_2} = \frac{l_1}{l_2}. \qquad (3)$$

From (1), (2), and (3) and § 164, it follows that when lengths, diameters, and forces are all different,

$$\frac{d_1}{d_2} = \frac{f_1}{f_2} \times \frac{l_1}{l_2} \times \frac{D_2^2}{D_1^2}. \qquad (4)$$

If a force of 1 pound will stretch an iron wire which is 200 centimeters long and .5 millimeter in diameter through 1 millimeter, what force is required to stretch an iron wire 150 centimeters long and 1.25 millimeters in diameter through .5 millimeter?

4. When the temperature of a gas is constant, its volume is found to be inversely proportional to the pressure to which the gas is subjected, *i.e.*, algebraically stated,

$$\frac{V_1}{V_2} = \frac{P_2}{P_1}. \qquad (5)$$

At the bottom of a lake 30 meters deep, where the pressure is 4000 grams per square centimeter, a bubble of air has a volume of 1 cubic centimeter as it escapes from a diver's suit. To what volume will it have expanded when it reaches the surface where the atmospheric pressure is about 1000 grams per square centimeter?

5. The electrical resistance of a wire varies directly as its length and inversely as its area. If a copper wire 1 centimeter

in diameter has a resistance of 1 unit per mile, how many units of resistance will a copper wire have which is 500 feet long and 3 millimeters in diameter?

6. The illumination from a source of light varies inversely as the square of the distance from the source. A book which is now 10 inches from the source is moved 15 inches farther away. How much will the light received be reduced?

7. The period of vibration of a pendulum is found to vary directly as the square root of its length. If a pendulum 1 meter long ticks seconds, what will be the period of vibration of a pendulum 30 centimeters long?

8. The force with which the earth pulls on any body outside of its surface is found to vary inversely as the square of the distance from its center. If the surface of the earth is 4000 miles from the center, what would a pound weight weigh 15000 miles from the earth?

9. The number of vibrations made per second by a guitar string of given diameter and material is inversely proportional to its length and directly proportional to the square root of the force with which it is stretched. If a string 3 feet long, stretched with a force of 20 pounds, vibrates 400 times per second, find the number of vibrations made by a string 1 foot long, stretched by a force of 40 pounds.

FACTORING

In Type V, § 103, we learned to transform certain trinomials into Type I, § 103. By means of the results of § 213, we are now able to extend this method to expressions not readily factored by the simpler processes.

222. Factoring of Quadratic Expressions.

A *quadratic expression* is an expression of the form

$$ax^2 + bx + c.$$

We have,

$$ax^2 + bx + c = a\left(x^2 + \frac{bx}{a} + \frac{c}{a}\right)$$

$$= a\left[x^2 + \frac{bx}{a} + \left(\frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right]$$

$$= a\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right),$$

by § 103, I.

But by § 213, the roots of $ax^2 + bx + c = 0$ are

$$-\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \text{ and } -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}.$$

Hence, to factor a quadratic expression place it equal to zero, and solve the equation thus formed.

Then the required factors are the coefficient of x^2 in the given expression, x minus the first root, and x minus the second.

Sometimes the expression may be written as the difference of two squares and the method of § 103, V, used.

Ex. Factor $x^4 + 1$.

$$x^4 + 1 = (x^4 + 2x^2 + 1) - 2x^2$$
$$= (x^2 + 1)^2 - (x\sqrt{2})^2$$
$$= (x^2 + x\sqrt{2} + 1)(x^2 - x\sqrt{2} + 1).$$

EXERCISE 49

Factor the following:

1. $x^2 + 11x + 24$.
2. $m^2 - m - 210$.
3. $3a^2 - 10a - 8$.
4. $2a^2 - 11a + 15$.
5. $28 - 13x - 6x^2$.
6. $8c^2 + 8c - 6$.
7. $9x^2 - 6x - 8$.
8. $16 + 18b - 9b^2$.
9. $2 + 5p - 25p^2$.
10. $49a^2 + 28a - 11$.
11. $a^4 + 4$.
12. $x^4 + y^4$.
13. $9d^4 + 22d^2 + 25$.
14. $16a^4 - 78a^2b^2 + 81b^4$.

15. $2x^2 - 5xy + 3y^2 + 5x - 7y + 2.$
16. $x^2 - xy - 2y^2 - 5x + y + 6.$
17. $a^2 + 2ab - 15b^2 - 3ac + 17bc - 4c^2.$
18. $3a^2 - 23ab + 14b^2 + a + 31b - 10.$
19. $6x^2 + 7xy + 2y^2 - 26x - 16y + 24.$
20. $10x^2 + xy - 24y^2 + 26x + 54y - 12.$

Solve the following equations:

21. $x^3 + 27 = 0.$
22. $x^4 - 20x^2 + 64 = 0.$
23. $x^4 + 2x^2 + 9 = 0.$
24. $x^4 + 4x^2 + 9 = 0.$
25. $(x+2)(3x^2+4x+5) = 0.$
26. $x^6 - 64 = 0.$
27. $2x^3 - 3x^2 + 4x - 6 = 0.$
28. $x^4 - 2x^3 + 5x^2 = 0.$
29. $(x^2-4)(3x^2+x-10) = 0.$
30. $x^7 - 729x = 0.$
31. $x^4 - 9x^2 + 14 = 0.$
32. $9x^4 - 2x^2 + 4 = 0.$
33. $x^4 - 16 = 0.$
34. $2x^3 + 6x^2 - 18x - 54 = 0.$
35. $(4x^2 - 1)(x^2 + x + 1) = 0.$

SIMULTANEOUS QUADRATIC EQUATIONS

223. In solving simultaneous quadratic equations involving two unknown numbers it is necessary to eliminate one of the unknowns as was done in simultaneous linear equations.

The elimination of an unknown number from two equations of the second degree will often produce an equation of the *fourth* degree with one unknown number which cannot be solved by the ordinary methods. The following general directions will lead to the solution of many types.

224. CASE I. *When each equation is in the form*

$$ax^2 + by^2 = c.$$

In this case, either x^2 or y^2 can be eliminated by addition or subtraction (§ 42, II, III).

CASE II. *When each equation is of the second degree, and homogeneous; that is, when each term involving the unknown numbers is of the second degree with respect to them* (§ 23).

The equations may then be solved as follows:

Ex. Solve the equations
$$\begin{cases} x^2 - 2xy = 5, & (1) \\ x^2 + y^2 = 29. & (2) \end{cases}$$

Dividing (1) by (2), $\frac{x^2 - 2xy}{x^2 + y^2} = \frac{5}{29}$, or $29x^2 - 58xy = 5x^2 + 5y^2$.

Then, $5y^2 + 58xy - 24x^2 = 0$, or $(5y - 2x)(y + 12x) = 0$.

Placing $5y - 2x = 0$, $y = \frac{2x}{5}$; substituting in (1),

$$x^2 - \frac{4x^2}{5} = 5, \text{ or } x^2 = 25.$$

Then, $x = \pm 5$, and $y = \frac{2x}{5} = \pm 2$.

Case III. *When the given equations are symmetrical with respect to x and y; that is, when x and y can be interchanged without changing the equation.*

Equations of this kind may be solved by combining them in such a way as to obtain the values of $x + y$ and $x - y$.

1. Solve the equations
$$\begin{cases} x + y = 2. & (1) \\ xy = -15. & (2) \end{cases}$$

Squaring (1),	$x^2 + 2xy + y^2 = 4.$	
Multiplying (2) by 4,	$4xy = -60.$	
Subtracting,	$x^2 - 2xy + y^2 = 64.$	
Extracting square roots,	$x - y = \pm 8.$	(3)
Adding (1) and (3),	$2x = 2 \pm 8 = 10$ or -6.	
Whence,	$x = 5$ or -3.	
Subtracting (3) from (1),	$2y = 2 \mp 8 = -6$ or 10.	
Whence,	$y = -3$ or 5.	

The solution is $x = 5$, $y = -3$; or, $x = -3$, $y = 5$.

The above method offers the most desirable form of solution and should be employed when possible.

If one equation is of the second degree, the other of the first degree, and they are not symmetrical, Case IV should be used.

CASE IV. *When one equation is of the second degree and the other of the first.*

Equations of this kind may be solved by finding one of the unknown numbers in terms of the other from the first degree equation, and substituting this value in the other equation.

Ex. Solve the equations $\begin{cases} 2x^2 - xy = 6y. & (1) \\ x + 2y = 7. & (2) \end{cases}$

From (2), $2y = 7 - x$, or $y = \dfrac{7-x}{2}$. (3)

Substituting in (1), $2x^2 - x\left(\dfrac{7-x}{2}\right) = 6\left(\dfrac{7-x}{2}\right)$.

Clearing of fractions, $4x^2 - 7x + x^2 = 42 - 6x$, or $5x^2 - x = 42$.

Solving, $x = 3$ or $-\dfrac{14}{5}$.

Substituting in (3), $y = \dfrac{7-3}{2}$ or $\dfrac{7+\frac{14}{5}}{2} = 2$ or $\dfrac{49}{10}$.

The solution is $x = 3$, $y = 2$; or $x = -\frac{14}{5}$, $y = \frac{49}{10}$.

Certain examples where one equation is of the *third* degree and the other of the first may be solved by the method of Case IV.

225. Special Methods for the Solution of Simultaneous Equations of Higher Degree.

No general rules can be given for examples which do not come under the cases just considered; various artifices are employed, familiarity with which can only be gained by experience.

1. Solve the equations $\begin{cases} x^3 - y^3 = 19. & (1) \\ x^2y - xy^2 = 6. & (2) \end{cases}$

Multiply (2) by 3, $3x^2y - 3xy^2 = 18$. (3)

Subtract (3) from (1), $x^3 - 3x^2y + 3xy^2 - y^3 = 1$.

Extracting cube roots, $x - y = 1$. (4)

Dividing (2) by (4), $xy = 6$. (5)

Solving equations (4) and (5) by the method of § 224, Case III, we find $x = 3$, $y = 2$; or, $x = -2$, $y = -3$.

2. Solve the equations $\begin{cases} x^3+y^3=9xy. \\ x+y=6. \end{cases}$

Putting $x = u + v$ and $y = u - v$,

$(u+v)^3 + (u-v)^3 = 9(u+v)(u-v)$, or, $2u^3 + 6uv^2 = 9(u^2 - v^2)$; (1)

and $\qquad (u+v) + (u-v) = 6,\ 2u = 6$, or $u = 3$.

Putting $u = 3$ in (1), $\quad 54 + 18v^2 = 9(9 - v^2)$.

Whence, $\qquad v^2 = 1$, or $v = \pm 1$.

Therefore, $\qquad x = u + v = 3 \pm 1 = 4$ or 2;

and $\qquad y = u - v = 3 \mp 1 = 2$ or 4.

The solution is $x = 4,\ y = 2$; or, $x = 2,\ y = 4$.

The artifice of substituting $u + v$ and $u - v$ for x and y is advantageous in any case where the given equations are *symmetrical* (§ 224, Case III) with respect to x and y. See also Ex. 4.

3. Solve the equations $\begin{cases} x^2+y^2+2x+2y=23. & (1) \\ xy=6. & (2) \end{cases}$

Multiplying (2) by 2, $\qquad 2xy = 12.$ (3)

Add (1) and (3), $\quad x^2 + 2xy + y^2 + 2x + 2y = 35.$

Or, $\qquad (x+y)^2 + 2(x+y) = 35.$

Completing the square, $(x+y)^2 + 2(x+y) + 1 = 36.$

Then, $\qquad (x+y) + 1 = \pm 6$; and $x + y = 5$ or -7. (4)

Squaring (4), $\qquad x^2 + 2xy + y^2 = 25$ or 49.

Multiplying (2) by 4, $\qquad 4xy \quad = 24.$

Subtracting, $\qquad x^2 - 2xy + y^2 = 1$ or 25.

Whence, $\qquad x - y = \pm 1$ or ± 5. (5)

Adding (4) and (5), $\qquad 2x = 5 \pm 1$, or -7 ± 5.

Whence, $\qquad x = 3,\ 2,\ -1$, or -6.

Subtracting (5) from (4), $\qquad 2y = 5 \mp 1$, or -7 ∓ 5.

Whence, $\qquad y = 2,\ 3,\ -6$, or -1.

The solution is $x = 3,\ y = 2$; $x = 2,\ y = 3$; $x = -1,\ y = -6$; or $x = -6$, $y = -1$.

4. Solve the equations $\begin{cases} x^4 + y^4 = 97. \\ x + y \ \ = -1. \end{cases}$

Putting $x = u + v$ and $y = u - v$,

$$(u+v)^4 + (u-v)^4 = 97, \text{ or } 2\,u^4 + 12\,u^2v^2 + 2\,v^4 = 97, \tag{1}$$

and $$(u+v) + (u-v) = -1,\ 2\,u = -1, \text{ or } u = -\tfrac{1}{2}.$$

Substituting value of u in (1), $\frac{1}{8} + 3\,v^2 + 2\,v^4 = 97.$

Solving this, $$v^2 = \frac{25}{4} \text{ or } -\frac{31}{4}; \text{ and } v = \pm\frac{5}{2} \text{ or } \pm\frac{\sqrt{-31}}{2}.$$

Then, $x = u + v = -\frac{1}{2} \pm \frac{5}{2}, \text{ or } -\frac{1}{2} \pm \frac{\sqrt{-31}}{2} = 2, -3, \text{ or } \frac{-1 \pm \sqrt{-31}}{2};$

and $$y = u - v = -\frac{1}{2} \mp \frac{5}{2}, \text{ or } -\frac{1}{2} \mp \frac{\sqrt{-31}}{2} = -3, 2, \text{ or } \frac{-1 \mp \sqrt{-31}}{2}.$$

The solution is $x = 2,\ y = -3;\ x = -3,\ y = 2;\ x = \frac{-1+\sqrt{-31}}{2},$ $y = \frac{-1-\sqrt{-31}}{2}$; or $x = \frac{-1-\sqrt{-31}}{2},\ y = \frac{-1+\sqrt{-31}}{2}.$

MISCELLANEOUS EXAMPLES

EXERCISE 50

Solve the following equations and verify each result:

1. $\begin{cases} 2\,xy + x = -36. \\ xy - 3\,y = -5. \end{cases}$

2. $\begin{cases} x^2 + y^2 + x - y = 32. \\ xy = 6. \end{cases}$

3. $\begin{cases} g^2 + h^2 = \frac{289}{36}. \\ gh = \frac{10}{3}. \end{cases}$

4. $\begin{cases} x^2 - 3\,xy - 4\,y^2 = 0. \\ 3\,x - 5\,y \ = 46. \end{cases}$

5. $\begin{cases} x^2 - 2\,y^2 + 3\,x = -8. \\ x^2 - 2\,y^2 - 4\,y = -2. \end{cases}$

6. $\begin{cases} \dfrac{1}{x^2} + \dfrac{1}{y^2} = 74. \\ \dfrac{1}{x} - \dfrac{1}{y} = 12. \end{cases}$

7. $\begin{cases} \dfrac{2\,x}{y} - 5\,x = \dfrac{11}{2}. \\ \dfrac{3\,y}{x} + 4\,y = \dfrac{2}{3}. \end{cases}$

8. $\begin{cases} \dfrac{x}{y} + \dfrac{y}{x} = -\dfrac{10}{3}. \\ x - y = 1. \end{cases}$

9. $\begin{cases} x - \dfrac{2}{y} = -\dfrac{a}{b}. \\ y + \dfrac{2}{x} = \dfrac{3b}{a}. \end{cases}$

10. $\begin{cases} x^4 + y^4 = 17. \\ x - y = 3. \end{cases}$

11. $\begin{cases} 4d + k - 3dk = -6. \\ d - 5k + 2dk = 10. \end{cases}$

12. $\begin{cases} \dfrac{1}{x^2} + \dfrac{1}{xy} + \dfrac{1}{y^2} = 49. \\ \dfrac{1}{x} + \dfrac{1}{y} = 8. \end{cases}$

13. $\begin{cases} x + y = 35. \\ \sqrt[3]{x} + \sqrt[3]{y} = 5. \end{cases}$

14. $\begin{cases} 11x^2 - xy - y^2 = 45. \\ 7x^2 + 3xy - 2y^2 = 20. \end{cases}$

15. $\begin{cases} x^2 + 4xy = 13. \\ 2xy + 9y^2 = 87. \end{cases}$

16. $\begin{cases} x^2y^2 - 24xy + 95 = 0. \\ 3x - 2y = -13. \end{cases}$

17. $\begin{cases} 3x^2 - 5xy = 2a^2 + 13ab - 7b^2. \\ x + y = 3(a - b). \end{cases}$

18. $\begin{cases} \dfrac{3x+2y}{3x-2y} + \dfrac{3x-2y}{3x+2y} = \dfrac{41}{20}. \\ 8y^2 + 3x^2 = 29. \end{cases}$

19. $\begin{cases} \dfrac{1}{xy} = 6a^2. \\ x + y = 5axy. \end{cases}$

20. $\begin{cases} \dfrac{1}{x^3} + \dfrac{1}{y^3} = -19a^3. \\ \dfrac{1}{x} + \dfrac{1}{y} = -a. \end{cases}$

21. $\begin{cases} e^2 + 9t^2 + 4e = 9. \\ et + 2t = -2. \end{cases}$

22. $\begin{cases} x^3 + y^3 = 2a^3 + 24a. \\ x^2y + xy^2 = 2a^3 - 8a. \end{cases}$

23. $\begin{cases} \sqrt{2x^2 - 9} = 3y + 6. \\ \sqrt{x^4 - 17y^2} = x^2 - 5. \end{cases}$

24. $\begin{cases} 3x^2 - xy - xz = 4. \\ 5x - 2y = 1. \\ 4x + 3z = -5. \end{cases}$

25. $\begin{cases} x^2y + xy^2 = 56. \\ x + y = -1. \end{cases}$

26. $\begin{cases} \dfrac{x^2}{y} + \dfrac{y^2}{x} = \dfrac{19}{6}. \\ \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{6}. \end{cases}$

27. $\begin{cases} 3x^2 + 3y^2 = 10xy. \\ \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{4}{3}. \end{cases}$

28. $\begin{cases} x^2y + y^2x = 42. \\ \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{7}{6}. \end{cases}$

29. $\begin{cases} 5q^2 + qs - 3s^2 = 27. \\ 4q^2 - 4qs + 3s^2 = 72. \end{cases}$

30. $\begin{cases} y^2 + 4xy - 3y = 42. \\ 2y^2 - xy + 5y = -10. \end{cases}$

31. $\begin{cases} 16x^2y^2 - 104xy = -105. \\ x - y = -2. \end{cases}$

32.* $\begin{cases} x^4 + x^2y^2 + y^4 = 481. \\ x^2 - xy + y^2 = 37. \end{cases}$

33. $\begin{cases} 9x^2 - 13xy - 3x = -123. \\ xy + 4y^2 + 2y = 125. \end{cases}$

* Divide the first equation by the second.

EXERCISE 51

1. Find two numbers whose product is 112 and whose difference is 6.

2. A rectangular field has a perimeter of 104 rods and an area of 4 acres. Find its dimensions.

3. The square of the sum of two numbers minus four times their product equals 49, and the difference of their squares equals 175. What are the numbers?

4. The sum of the cubes of two numbers is 855; and if the sum of the numbers be multiplied by their product, the result will be 840. What are the numbers?

5. There is a number consisting of two digits, the sum of whose squares is 80; and if the sum of the digits be multiplied by 4, the number will be expressed with its digits reversed. What is the number?

6. A man loaned a sum of money at 6% for a given time and received \$240 interest; if he had loaned the same sum for two years longer at the rate represented by the first number of years, he would have received \$40 more than at first. Find the time and the amount loaned.

7. If 5 be added to the denominator and subtracted from the numerator of a certain fraction, it will be expressed by its reciprocal; and the difference of the squares of numerator and denominator equals 65. What is the fraction?

8. A number consists of three digits, the second of which is twice the first. The sum of the squares of the digits equals

89, and if 99 be subtracted from the number, the digits will be reversed. What is the number?

9. A man buys two pieces of cloth, each containing as many yards as its price per yard in cents, and he pays \$41 for the whole amount. If the prices for the two pieces of cloth had been interchanged, his bill would have been \$1 less. How many yards of each did he buy and what was the price per yard?

10. Two squares have together an area of 613 square rods. If the side of the first square were decreased by 6, and that of the second increased by 1, their perimeters would be in the ratio of 2 to 3. Find the side of each square.

11. There are two numbers whose sum decreased by the square root of their product is 13; and the sum of their squares increased by their product is 481. Find the numbers.

12. Two boys count their pennies. They find that the product of the numbers representing them is 84, and that the square of their sum decreased by twice their difference is 351. How many did each have?

13. There are two numbers whose difference is 819, and the difference of their cube roots is 3. What are the numbers?

14. There is a difference of one hour's time in two trains which go from A to B, the rate of the first train being 5 miles an hour more than that of the second train. If the speed of each train were increased 2 miles per hour, the difference in time from A to B would be decreased 7 minutes 55 seconds. Find the distance from A to B and the rate of each train.

15. The difference of the perimeters of a square and a circle is 5.752 feet and the circle contains 81.86 square feet more than the square. Find the radius of the circle and the side of the square.

16. In an isosceles triangle the product of the base and one leg is 168, and the difference between the squares of the base and leg is 52. Find the altitude of the triangle.

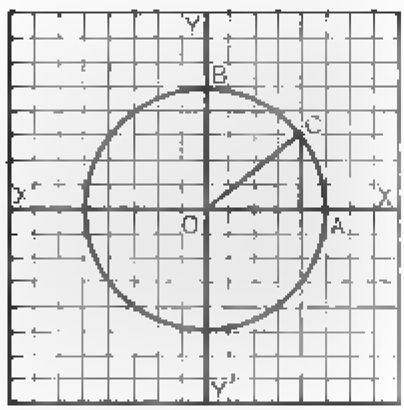
17. The perimeter of a rectangle is 46 inches. If its length be increased 3 inches, its area will be 153 square inches. Find its dimensions. Is there more than one such rectangle? Explain.

18. If the sum of the denominator and numerator of a certain fraction be divided by their difference, the quotient is 9. But if the product of the numerator and denominator be divided by their sum, the quotient is 2 with a remainder of 2. Find the fraction. What principle of proportion is illustrated in this problem? If this principle is applied, are simultaneous equations necessary?

226. It was noted in §§ 224, 225, that two second degree equations had four solutions, or pairs of values for x and y, that a second degree and a first degree equation had two solutions, that if imaginary roots entered they were always in pairs. The geometric explanation for this is readily seen if the equations are plotted.

Ex. 1. Consider the equation $x^2+y^2=25$.

This means that, for any point on the graph, the square of the abscissa, plus the square of the ordinate, equals 25.



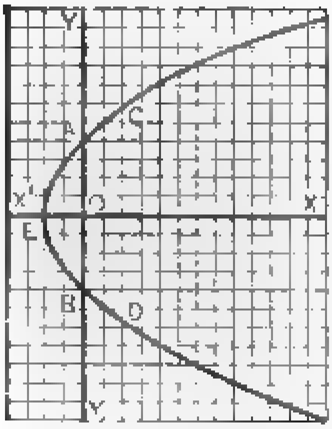
But the square of the abscissa of any point, plus the square of the ordinate, equals the square of the distance of the point from the origin; for the distance is the hypotenuse of a right triangle, whose other two sides are the abscissa and ordinate.

Then the square of the distance from O of any point on the graph is 25; or, the distance from O of any point on the graph is 5.

Thus, the graph is a circle of radius 5, having its center at O.

(The graph of any equation of the form $x^2+y^2=a$ is a circle.)

Ex. 2. Consider the equation $y^2 = 4x + 4$.



If $x = 0$, $y^2 = 4$, or $y = \pm 2$. (A, B)

If $x = 1$, $y^2 = 8$, or $y = \pm 2\sqrt{2}$. (C, D)

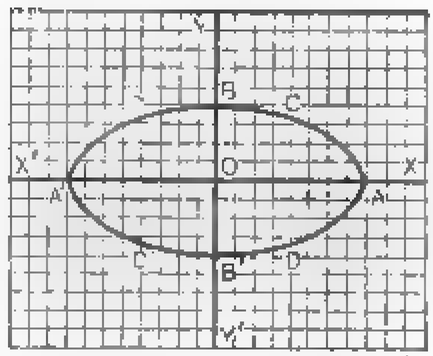
If $x = -1$, $y = 0$, Etc. (E)

The graph extends indefinitely to the right of YY'.

If x is negative and < -1, y^2 is negative, and therefore y *imaginary;* then, no part of the graph lies to the left of E.

(The graph of Ex. 2 is a *parabola;* as also is the graph of any equation of the form $y^2 = ax$ or $y^2 = ax + b$.)

Ex. 3. Consider the equation $x^2 + 4y^2 = 4$.



In this case it is convenient to first locate the points where the graph intersects the axes.

If $y = 0$, $x^2 = 4$, or $x = \pm 2$. (A, A')

If $x = 0$, $4y^2 = 4$, or $y = \pm 1$. (B, B')

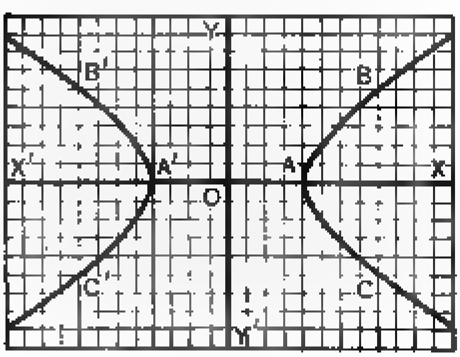
Putting $x = \pm 1$, $4y^2 = 3$, $y^2 = \frac{3}{4}$, or $y = \pm\frac{\sqrt{3}}{2}$. (C, D, C', D')

If x has any value > 2, or < -2, y^2 is negative, and y imaginary; then, no part of the graph lies to the right of A, or left of A'.

If y has any value > 1, or < -1, x^2 is negative, and x imaginary; then, no part of the graph lies above B, or below B'.

(The graph of Ex. 3 is an *ellipse;* as also is the graph of any equation of the form $ax^2 + by^2 = c$.)

Ex. 4. Consider the equation $x^2 - 2y^2 = 1$.



Here $x^2 - 1 = 2y^2$, or $y^2 = \frac{x^2 - 1}{2}$.

If $x = \pm 1$, $y^2 = 0$, or $y = 0$. (A', A)

If x has any value between 1 and -1, y^2 is negative, and y imaginary.

Then, no part of the graph lies between A and A'.

If $x = \pm 2$, $y^2 = \frac{3}{2}$, or $y = \pm\sqrt{\frac{3}{2}}$. (B, C, B', C')

educe $\dfrac{(e^x+e^{-x})^2-(e^x-e^{-x})^2}{(e^x+e^{-x})^2}$ to the form $\left(\dfrac{2}{e^x+e^{-x}}\right)^2$.

educe $\dfrac{1}{\sqrt{2ax-x^2}}$ to the form $\dfrac{1}{a\sqrt{2\left(\dfrac{x}{a}\right)-\left(\dfrac{x}{a}\right)^2}}$.

Reduce $\dfrac{(x^{\frac{m}{n}}+x^{-\frac{m}{n}})^2-4}{x^{\frac{2m}{n}}-x^{-\frac{2m}{n}}}$ to the form $\dfrac{x^{\frac{m}{n}}-x^{-\frac{m}{n}}}{x^{\frac{m}{n}}+x^{-\frac{m}{n}}}$.

Reduce $\dfrac{1+\dfrac{x}{\sqrt{x^2-a^2}}}{x+\sqrt{x^2-a^2}}+\dfrac{\dfrac{1}{a}}{\dfrac{x}{a}\sqrt{\dfrac{x^2-a^2}{a^2}}}$ to the form $\dfrac{1}{x}\sqrt{\dfrac{x+a}{x-a}}$.

Reduce $\dfrac{x[1+x(x^2+y^2)^{-\frac{1}{2}}]}{x+\sqrt{x^2+y^2}}+\dfrac{y^2(x^2+y^2)^{-\frac{1}{2}}}{x+\sqrt{x^2+y^2}}$ to unity.

Reduce $\dfrac{\dfrac{ax}{(a^2-x^2)^{\frac{3}{2}}}}{\dfrac{a}{(a^2-x^2)^{\frac{1}{2}}}\sqrt{\left(\dfrac{a}{\sqrt{a^2-x^2}}\right)^2-1}}$ to the form $\dfrac{1}{\sqrt{a^2-x^2}}$.

$S=\sqrt{1+K^2}$. If $K=\dfrac{\sqrt{2ay-y^2}}{y}$, find $S=\sqrt{\dfrac{2a}{y}}$.

). Reduce $\dfrac{1}{\sqrt{1-3x-x^2}}$ to the form $\dfrac{\dfrac{2}{\sqrt{13}}}{\sqrt{1-\left(\dfrac{3+2x}{\sqrt{13}}\right)^2}}$.

1. Reduce $\dfrac{x^3}{x^8-x^4-6}$ to the form $\dfrac{1}{25}\left[\dfrac{-4x^3}{1-\left(\dfrac{1-2x^4}{5}\right)^2}\right]$.

12. Reduce $\sqrt{2ax-x^2}$ to the form

$$\frac{a}{\sqrt{1-\left(\dfrac{x-a}{a}\right)^2}}-\frac{(x-a)^2}{a\sqrt{1-\left(\dfrac{x-a}{a}\right)^2}}.$$

EXERCISE 52

Find the graphs of the following sets of equations, and in each case verify the principle of § 227:

1. $\begin{cases} x^2 + 4y^2 = 4. \\ x - y = 1. \end{cases}$

2. $\begin{cases} x^2 - 4y = -7. \\ 2x + 3y = 4. \end{cases}$

3. $\begin{cases} 9x^2 + y^2 = 148. \\ xy = -8. \end{cases}$

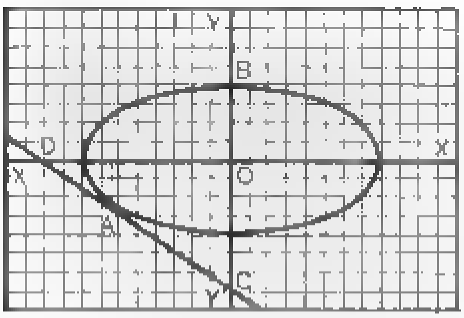
4. $\begin{cases} x^2 + y^2 = 29. \\ xy = 10. \end{cases}$

5. $\begin{cases} 2x^2 + 5y^2 = 53. \\ 3x^2 - 4y^2 = -24. \end{cases}$

6. $\begin{cases} x^2 + y^2 = 13. \\ 4x - 9y = 6. \end{cases}$

228. *Ex.* 1. Consider the equations

$$\begin{cases} x^2 + 4y^2 = 4, & (1) \\ 2x + 3y = -5. & (2) \end{cases}$$



Scale: $\frac{1}{4}$ inch.

The graph of $x^2 + 4y^2 = 4$ is the ellipse AB.

The graph of $2x + 3y = -5$ is the straight line CD.

If y or x is eliminated between these two equations, we find that the resulting equation containing one unknown number is such that if all the terms are transposed to one member, that member is a trinomial perfect square. Hence, the equation has equal roots and the line and curve are tangent at A (218, II).

If in Ex. 1, § 228, the second equation had been $2x + 3y = -10$ (2), the roots would have been imaginary and the line would not have met the ellipse.

REVIEW EXAMPLES

EXERCISE 53

1. Reduce $\dfrac{\sqrt{1-x^2} + \dfrac{x^2}{\sqrt{1-x^2}}}{\sqrt{1-x^2}}$ to the form $\dfrac{1}{1-x^2}$.

2. Reduce $\dfrac{\dfrac{a}{x^2}}{1 + \dfrac{a^2}{x^2}} + \dfrac{\dfrac{a}{(x-a)^2}}{\dfrac{x+a}{x-a}}$.

3. Reduce $\dfrac{(e^x+e^{-x})^2-(e^x-e^{-x})^2}{(e^x+e^{-x})^2}$ to the form $\left(\dfrac{2}{e^x+e^{-x}}\right)^2$.

4. Reduce $\dfrac{1}{\sqrt{2ax-x^2}}$ to the form $\dfrac{1}{a\sqrt{2\left(\dfrac{x}{a}\right)-\left(\dfrac{x}{a}\right)^2}}$.

5. Reduce $\dfrac{(x^{\frac{m}{n}}+x^{-\frac{m}{n}})^2-4}{x^{\frac{2m}{n}}-x^{-\frac{2m}{n}}}$ to the form $\dfrac{x^{\frac{m}{n}}-x^{-\frac{m}{n}}}{x^{\frac{m}{n}}+x^{-\frac{m}{n}}}$.

6. Reduce $\dfrac{1+\dfrac{x}{\sqrt{x^2-a^2}}}{x+\sqrt{x^2-a^2}}+\dfrac{\dfrac{1}{a}}{\dfrac{x}{a}\sqrt{\dfrac{x^2-a^2}{a^2}}}$ to the form $\dfrac{1}{x}\sqrt{\dfrac{x+a}{x-a}}$.

7. Reduce $\dfrac{x[1+x(x^2+y^2)^{-\frac{1}{2}}]}{x+\sqrt{x^2+y^2}}+\dfrac{y^2(x^2+y^2)^{-\frac{1}{2}}}{x+\sqrt{x^2+y^2}}$ to unity.

8. Reduce $\dfrac{\dfrac{ax}{(a^2-x^2)^{\frac{3}{2}}}}{\dfrac{a}{(a^2-x^2)^{\frac{1}{2}}}\sqrt{\left(\dfrac{a}{\sqrt{a^2-x^2}}\right)^2-1}}$ to the form $\dfrac{1}{\sqrt{a^2-x^2}}$.

9. $S=\sqrt{1+K^2}$. If $K=\dfrac{\sqrt{2ay-y^2}}{y}$, find $S=\sqrt{\dfrac{2a}{y}}$.

10. Reduce $\dfrac{1}{\sqrt{1-3x-x^2}}$ to the form $\dfrac{\dfrac{2}{\sqrt{13}}}{\sqrt{1-\left(\dfrac{3+2x}{\sqrt{13}}\right)^2}}$.

11. Reduce $\dfrac{x^3}{x^8-x^4-6}$ to the form $\dfrac{1}{25}\left[\dfrac{-4x^3}{1-\left(\dfrac{1-2x^4}{5}\right)^2}\right]$.

12. Reduce $\sqrt{2ax-x^2}$ to the form

$$\frac{a}{\sqrt{1-\left(\dfrac{x-a}{a}\right)^2}}-\frac{(x-a)^2}{a\sqrt{1-\left(\dfrac{x-a}{a}\right)^2}}.$$

13. $2\tan x + (\tan x)^2 - 3 = 0$; find $\tan x$.

14. $2\cos x + \dfrac{1}{\cos x} = 3$; find $\cos x$.

15. $\dfrac{1}{\cot x} + 2\cot x = \dfrac{5}{2}\sqrt{1+\cot^2 x}$; find $\cot x$.

16. Reduce $\dfrac{n(1+x)^n \cdot x^{n-1} - n(1+x)^{n-1} \cdot x^n}{(1+x)^{2n}}$ to $\dfrac{nx^{n-1}}{(1+x)^{n+1}}$.

17. $S = \dfrac{1}{3\sqrt{3}}(12+x)^{\frac{3}{2}} - 8$. Evaluate S when $x = 15$.

18. Reduce $\dfrac{-x^2\left(\dfrac{4x^3}{2\sqrt{1-x^4}}\right) - 2x - 2x\sqrt{1-x^4}}{x^4}$ to the form

$$-\frac{2}{x^3}\left(1 + \frac{1}{\sqrt{1-x^4}}\right).$$

19. Reduce $\dfrac{2 + \dfrac{2x-1}{\sqrt{x^2-x-1}}}{2x-1+2\sqrt{x^2-x-1}}$ to the form $\dfrac{1}{\sqrt{x^2-x-1}}$.

20. Reduce $\dfrac{\left(\dfrac{x-2}{x+2}\right)^{\frac{3}{4}} + \dfrac{3}{4}\left(\dfrac{x-2}{x+2}\right)^{-\frac{1}{4}}\left(\dfrac{2}{x+2}\right)^2}{\left(\dfrac{x-2}{x+2}\right)^{\frac{3}{4}}}$ to the form $\dfrac{x^2-1}{x^2-4}$.

21. Reduce $\dfrac{\left[1 + \left(\dfrac{e^{\frac{x}{a}} - e^{-\frac{x}{a}}}{2}\right)^2\right]\left(\dfrac{e^{\frac{x}{a}} - e^{-\frac{x}{a}}}{2}\right)}{\dfrac{e^{\frac{x}{a}} + e^{-\frac{x}{a}}}{2a}}$ to the form

$$\frac{a\left(e^{\frac{2x}{a}} - e^{-\frac{2x}{a}}\right)}{4}.$$

22. Reduce $y+\dfrac{1+\left(\dfrac{\sqrt{2ry-y^2}}{y}\right)^2}{\dfrac{-r}{y^2}}$ to the form $-y$.

23. Reduce $x-\dfrac{\left(1+\dfrac{4p^2}{y^2}\right)\dfrac{2p}{y}}{\dfrac{-4p^2}{y^3}}$ to $3x+2p$ when $y^2=4px$.

24. Reduce $-\dfrac{b^2}{a^2}\left[\dfrac{y-x\left(\dfrac{-b^2x}{a^2y}\right)}{y^2}\right]$ to $-\dfrac{b^4}{a^2y^3}$ when $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

IX. SERIES

ARITHMETIC PROGRESSION

229. An **Arithmetic Progression** is a series of terms in which each term, after the first, is obtained by adding to the preceding term a constant number called the *Common Difference*.

Thus, 1, 3, 5, 7, 9, 11, ⋯ is an arithmetic progression in which the common difference is 2.

Again, 12, 9, 6, 3, 0, -3, ⋯ is an arithmetic progression in which the common difference is -3.

An Arithmetic Progression is also called an *Arithmetic Series.*

230. *Given the first term, a, the common difference, d, and the number of terms, n, to find the last term, l.*

The progression is a, $a+d$, $a+2d$, $a+3d$, ⋯.

We observe that the coefficient of d in any term is less by 1 than the number of the term.

Then, in the nth term the coefficient of d will be $n-1$.

That is, $$l=a+(n-1)d. \qquad \text{(I)}$$

231. *Given the first term, a, the last term, l, and the number of terms, n, to find the sum of the terms, S.*

$$S = a + (a + d) + (a + 2\,d) + \cdots + (l - d) + l.$$

Writing the terms in reverse order,

$$S = l + (l - d) + (l - 2\,d) + \cdots + (a + d) + a.$$

Adding these equations term by term,

$$2\,S = (a + l) + (a + l) + (a + l) + \cdots + (a + l) + (a + l).$$

Therefore, $2\,S = n(a + l)$, and $S = \frac{n}{2}(a + l)$. (II)

232. Substituting in (II) the value of l from (I), we have

$$S = \frac{n}{2}[2\,a + (n - 1)d].$$

Ex. In the progression 8, 5, 2, -1, -4, $\cdots$, to 27 terms, find the last term and the sum.

Here, $a = 8$, $d = 5 - 8 = -3$, $n = 27$.
Substitute in (I), $l = 8 + (27 - 1)(-3) = 8 - 78 = -70$.
Substitute in (II), $S = \frac{27}{2}(8 - 70) = 27(-31) = -837$.

The common difference may be found by subtracting the first term from the second, or any term from the next following term.

EXERCISE 54

In each of the following find the last term and then the sum:

1. 2, 5, 8, $\cdots$, to 17 terms.
2. 3, 9, 15, $\cdots$, to 12 terms.
3. 7, 5, 3, $\cdots$, to 24 terms.
4. 1, $\frac{1}{2}$, 0, $\cdots$, to 32 terms.
5. $-\frac{1}{3}$, $-\frac{1}{12}$, $\frac{1}{6}$, $\cdots$, to 9 terms.
6. a, $a - 3\,b$, $a - 6b$, $\cdots$, to 15 terms.
7. $2\,x + 5\,y$, $x + 4\,y$, $3\,y$, $\cdots$, to 13 terms.

8. $\frac{2c-5d}{3}, \frac{c-4d}{6}, \frac{d-c}{3}, \cdots$, to 20 terms.

9. $\frac{3}{5x}, \frac{1}{10x}, -\frac{2}{5x}, \cdots$, to 19 terms.

10. $\frac{1}{25}, \frac{7}{50}, \frac{6}{25}, \cdots$, to 47 terms.

233. The *first term, common difference, number of terms, last term,* and *sum of the terms* are called the *elements* of the progression.

If any three of the five elements of an arithmetic progression are given, the other two may be found by substituting the known values in the fundamental formulæ (I) and (II), and solving the resulting equations.

1. Given $a = -\frac{5}{3}, n = 20, S = -\frac{5}{3}$; find d and l.

Substituting the given values in (II),

$$-\tfrac{5}{3} = 10(-\tfrac{5}{3} + l) \text{ or } -\tfrac{1}{6} = -\tfrac{5}{3} + l; \text{ then, } l = \tfrac{5}{3} - \tfrac{1}{6} = \tfrac{3}{2}.$$

Substituting the values of a, n, and l in (I), $\frac{3}{2} = -\frac{5}{3} + 19d$.

Whence, $\quad 19d = \frac{3}{2} + \frac{5}{3} = \frac{19}{6}$, and $d = \frac{1}{6}$.

2. Given $d = -3, l = -39, S = -264$; find a and n.

Substituting in (I), $-39 = a + (n-1)(-3)$, or $a = 3n - 42$. (1)

Substituting the values of l, S, and a in (II),

$$-264 = \frac{n}{2}(3n - 42 - 39), \text{ or } -528 = 3n^2 - 81n, \text{ or } n^2 - 27n + 176 = 0.$$

Whence, $$n = \frac{27 \pm \sqrt{729 - 704}}{2} = \frac{27 \pm 5}{2} = 16 \text{ or } 11.$$

Substituting in (1), $a = 48 - 42$ or $33 - 42 = 6$ or -9.

The solution is $a = 6, n = 16$; or, $a = -9, n = 11$.

The significance of the two answers is as follows:

If $a = 6$ and $n = 16$, the progression is $6, 3, 0, -3, -6, -9, -12, -15, -18, -21, -24, -27, -30, -33, -36, -39$.

If $a = -9$ and $n = 11$, the progression is

$$-9, -12, -15, -18, -21, -24, -27, -30, -33, -36, -39.$$

In each of these the sum is -264.

3. Given $a=\frac{1}{3}$, $d=-\frac{1}{12}$, $S=-\frac{3}{2}$; find l and n.

Substituting in (I), $l=\frac{1}{3}+(n-1)\left(-\frac{1}{12}\right)=\frac{5-n}{12}$. (1)

Substituting the values of a, S, and l in (II),

$$-\frac{3}{2}=\frac{n}{2}\left(\frac{1}{3}+\frac{5-n}{12}\right), \text{ or } -3=n\left(\frac{9-n}{12}\right), \text{ or } n^2-9n-36=0.$$

Whence, $$n=\frac{9\pm\sqrt{81+144}}{2}=\frac{9\pm 15}{2}=12 \text{ or } -3.$$

The value $n=-3$ must be rejected, for the number of terms in a progression must be a *positive integer*.

Substituting $n=12$ in (1), $l=\frac{5-12}{12}=-\frac{7}{12}$.

A *negative* or *fractional* value of n must be rejected, together with all other values dependent on it.

EXERCISE 55

1. Given $a=5$, $d=2$, $l=65$; find n and S.
2. Given $d=-3$, $n=42$, $l=-119$; find a and S.
3. Given $d=\frac{5}{6}$, $n=16$, $S=\frac{316}{3}$; find a and l.
4. Given $n=19$, $l=\frac{-59}{7}$, $S=-\frac{1045}{14}$; find a and d.
5. Given $S=-540$, $a=-23$, $n=48$; find l and d.
6. Given $d=-4$, $l=\frac{-3325}{64}$, $S=\frac{-11627}{32}$; find a and n.
7. Given $d=a-1$, $a=2a+5$, $S=44a+12$; find l and n.
8. Given $a=-8a$, $l=8a-16b$, $S=-136b$; find n and d.
9. Given $a=.4$, $l=34.6$, $n=20$; find d and S.
10. Given $S=18.15$, $d=.02$, $a=.23$; find l and n.
11. Given $S=\frac{-209}{3}$, $d=\frac{-3}{5}$, $n=15$; find a and l.
12. Given $n=26$, $d=\frac{7}{8}$, $l=\frac{-325}{8}$; find S and a.

234. From (I) and (II), *general formulæ* for the solution of examples like the above may be readily derived.

Ex. Given a, d, and S; derive the formula for n.

By § 232, $2S = n[2a + (n-1)d]$, or $dn^2 + (2a - d)n = 2S$.

This is a quadratic in n, and may be solved by the method of § 213; multiplying by $4d$, and adding $(2a - d)^2$ to both members,

$$4d^2n^2 + 4d(2a - d)n + (2a - d)^2 = 8dS + (2a - d)^2.$$

Extracting square roots, $2dn + 2a - d = \pm\sqrt{8dS + (2a - d)^2}$.

Whence, $$n = \frac{d - 2a \pm \sqrt{8dS + (2a - d)^2}}{2d}.$$

EXERCISE 56

1. Given a, l, and n; derive the formula for d.
2. Given a, n, and S; derive the formulæ for d and l.
3. Given d, n, and S; derive the formulæ for a and l.
4. Given a, d, and l; derive the formulæ for n and S.
5. Given d, l, and n; derive the formulæ for a and S.
6. Given l, n, and S; derive the formulæ for a and d.
7. Given a, d, and S; derive the formulæ for l.
8. Given a, l, and S; derive the formulæ for d and n.
9. Given d, l, and S; derive the formulæ for a and n.

235. Arithmetic Means.

We define *inserting m arithmetic means between two given numbers, a and b,* as finding an arithmetic progression of $m + 2$ terms, whose first and last terms are a and b.

Ex. Insert 5 arithmetic means between 3 and -5.

We find an arithmetic progression of 7 terms, in which $a = 3$, and $l = -5$; substituting $n = 7$, $a = 3$, and $l = -5$ in (I),

$$-5 = 3 + 6d, \text{ or } d = -\tfrac{4}{3}.$$

The progression is $3, \frac{5}{3}, \frac{1}{3}, -1, -\frac{7}{3}, -\frac{11}{3}, -5$.

236. Let x denote the arithmetical mean between a and b.

Then, $$x-a=b-x, \text{ or } 2x=a+b.$$

Whence, $$x=\frac{a+b}{2}.$$

That is, *the arithmetic mean between two numbers equals one-half their sum.*

EXERCISE 57

1. Insert 6 arithmetic means between 3 and 24.
2. Insert 12 arithmetic means between -5 and 73.
3. Insert 20 arithmetic means between $\frac{4}{5}$ and $-\frac{58}{15}$.
4. Insert 13 arithmetic means between $-\frac{1}{3}$ and $-\frac{23}{6}$.
5. Find the arithmetic mean between a^2-2a-9 and a^2-6a+1.
6. If $n-2$ arithmetic means are inserted between a and l, find the 4th term.

GEOMETRIC PROGRESSION

237. A **Geometric Progression** is a series of terms in which each term, after the first, is obtained by multiplying the preceding term by a constant number called the *Ratio*.

Thus, 2, 6, 18, 54, 162, ⋯ is a geometric progression in which the ratio is 3.

9, 3, 1, $\frac{1}{3}$, $\frac{1}{9}$, ⋯ is a geometric progression in which the ratio is $\frac{1}{3}$.

-3, 6, -12, 24, -48, ⋯ is a geometric progression in which the ratio is -2.

A Geometric Progression is also called a *Geometric Series*.

238. *Given the first term, a, the ratio, r, and the number of terms, n, to find the last term, l.*

The progression is $a, ar, ar^2, ar^3, \cdots$.

We observe that the exponent of r in any term is less by 1 than the number of the term.

Then, in the nth term the exponent of r will be $n-1$.

That is, $$l=ar^{n-1}. \qquad \text{(I)}$$

239. *Given the first term, a, the last term, l, and the ratio, r, to find the sum of the terms, S.*

$$S = a + ar + ar^2 + \cdots + ar^{n-3} + ar^{n-2} + ar^{n-1}. \qquad (1)$$

Multiplying each term by r,

$$rS = ar = ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1} + ar^n. \qquad (2)$$

Subtracting (1) from (2), $rS - S = ar^n - a$, or $S = \dfrac{ar^n - a}{r - 1}$.

But by (I), § 238, $rl = ar^n$.

Therefore, $$S = \frac{rl - a}{r - 1}. \qquad \text{(II)}$$

The *first term*, *ratio*, *number of terms*, *last term*, and *sum of the terms* are called the *elements* of the progression.

240. Examples.

1. In the progression 3, 1, $\frac{1}{3}$, ⋯, to 7 terms, find the last term and the sum.

Here, $a = 3,\ r = \frac{1}{3},\ n = 7.$

Substituting in (I), $l = 3\left(\dfrac{1}{3}\right)^6 = \dfrac{1}{3^5} = \dfrac{1}{243}.$

Substituting in (II), $S = \dfrac{\frac{1}{3} \times \frac{1}{243} - 3}{\frac{1}{3} - 1} = \dfrac{\frac{1}{729} - 3}{-\frac{2}{3}} = \dfrac{-\frac{2186}{729}}{-\frac{2}{3}} = \dfrac{1093}{243}.$

The ratio may be found by dividing the second term by the first, or any term by the next preceding term.

2. In the progression -2, 6, -18, ⋯, to 8 terms, find the last term and the sum.

Here, $a = -2,\ r = \dfrac{6}{-2} = -3,\ n = 8$; therefore,

$$l = -2(-3)^7 = -2 \times (-2187) = 4374.$$

$$S = \frac{-3 \times 4374 - (-2)}{-3 - 1} = \frac{-13122 + 2}{-4} = 3280.$$

EXERCISE 58

Find the last term and sum of the following:

1. $1, 3, 9, \cdots$ to 8 terms.
2. $2, 1, \frac{1}{2}, \cdots$ to 11 terms.
3. $5, -10, 20, \cdots$ to 12 terms.
4. $-\frac{1}{5}, \frac{2}{15}, -\frac{4}{45}, \cdots$ to 7 terms.
5. $\frac{3}{4}, \frac{1}{2}, \frac{1}{3}, \cdots$ to 6 terms.
6. $-\frac{7}{8}, -\frac{7}{10}, -\frac{14}{25}, \cdots$ to 8 terms.

241. If any three of the five elements of a geometric progression are given, the other two may be found by substituting the given values in the fundamental formulæ (I) and (II), and solving the resulting equations.

But in certain cases the operation involves the solution of an equation of a degree higher than the second; and in others the unknown number appears as an exponent, the solution of which form of equation can usually only be effected by the aid of logarithms (§ 110).

1. Given $a = -2$, $n = 5$, $l = -32$; find r and S.

Substituting the given values in (I), we have

$$-32 = -2r^4; \text{ whence, } r^4 = 16, \text{ or } r = \pm 2.$$

Substituting in (II),

If $\quad r = 2, \quad S = \dfrac{2(-32)-(-2)}{2-1} = -64 + 2 = -62.$

If $\quad r = -2, \quad S = \dfrac{(-2)(-32)-(-2)}{-2-1} = \dfrac{64+2}{-3} = -22.$

The solution is $r = 2$, $S = -62$; or, $r = -2$, $S = -22$.

The interpretation of the two answers is as follows:

If $r = 2$, the progression is $-2, -4, -8, -16, -32$, whose sum is -62.

If $r = -2$, the progression is $-2, 4, -8, 16, -32$, whose sum is -22.

2. Given $a=3$, $r=-\frac{1}{3}$, $S=\frac{1640}{729}$; find n and l.

Substituting in (II), $\dfrac{1640}{729}=\dfrac{-\frac{1}{3}l-3}{-\frac{1}{3}-1}=\dfrac{l+9}{4}$.

Whence, $l+9=\frac{6560}{729}$; or, $l=\frac{6560}{729}-9=-\frac{1}{729}$.

Substituting the values of a, r, and l in (I),

$$-\tfrac{1}{729}=3(-\tfrac{1}{3})^{n-1};\text{ or, }(-\tfrac{1}{3})^{n-1}=-\tfrac{1}{2187}.$$

Whence, by inspection, $n-1=7$, or $n=8$.

From (I) and (II) general formulæ may be derived for the solution of cases like the above.

If the given elements are n, l, and S, *equations* for a and r may be found, but there are no definite *formulæ* for their values.

The same is the case when the given elements are a, n, and S.

The general formulæ for n involve logarithms; these cases are discussed in § 110.

EXERCISE 59

1. Given $r=2$, $n=12$, $S=4095$; find a and l.
2. Given $a=2$, $r=-3$, $l=1458$; find n and S.
3. Given $l=-\frac{1}{800}$, $a=-\frac{16}{25}$, $n=10$; find r and S.
4. Given $a=\frac{7}{8}$, $l=3584$, $S=\frac{38227}{8}$; find r and n.
5. Given $r=\frac{1}{3}$, $n=5$, $l=\frac{1}{135}$; find a and S.
6. Given $S=-\frac{4681}{64}$, $a=-64$, $r=\frac{1}{8}$; find n and l.
7. Given a, l, and S; derive the formula for r.
8. Given r, l, and n; derive the formulæ for a and S.
9. Given a, n, and l; derive the formulæ for r and S.
10. Given S, n, and r; derive the formulæ for a and l.

242. Sum of a Geometric Progression to Infinity.

The limit (§ 125) to which the sum of the terms of a *decreasing* geometric progression approaches, when the number of

terms is indefinitely increased, is called the ***sum of the series to infinity.***

Formula (II), § 239, may be written

$$S = \frac{a - rl}{1 - r}.$$

It is evident that, by sufficiently continuing a decreasing geometric progression, the absolute value of the last term may be made less than any assigned number, however small.

Hence, when the number of terms is indefinitely increased, l, and therefore rl, approaches the limit 0.

Then, the fraction $\frac{a - rl}{1 - r}$ approaches the limit $\frac{a}{1 - r}$.

Therefore, the sum of a decreasing geometric progression to infinity is given by the formula

$$S = \frac{a}{1 - r}. \qquad \text{(III)}$$

Ex. Find the sum of the series 4, $-\frac{8}{3}$, $\frac{16}{9}$, $\cdots$ to infinity.

Here $a = 4$, $r = -\frac{2}{3}$.

Substituting in (III), $S = \frac{4}{1 + \frac{2}{3}} = \frac{12}{5}$.

To find the value of a repeating decimal.

This is a case of finding the sum of a decreasing geometric series to infinity, and may be solved by formula (III).

Ex. Find the value of .85151⋯.

We have, $.85151 \cdots = .8 + .051 + .00051 + \cdots$.

The terms after the first constitute a decreasing geometric progression in which $a = .051$, and $r = .01$.

Substituting in (III), $S = \frac{.051}{1 - .01} = \frac{.051}{.99} = \frac{51}{990} = \frac{17}{330}$.

Then the value of the given decimal is $\frac{8}{10} + \frac{17}{330}$, or $\frac{281}{330}$.

EXERCISE 60

Find the sum to infinity of the following:

1. $2, \frac{2}{3}, \frac{2}{9}, \cdots$.
2. $1, -\frac{1}{2}, \frac{1}{4}, \cdots$.
3. $\frac{5}{6}, \frac{1}{6}, \frac{1}{30}, \cdots$.
4. $-\frac{8}{9}, -\frac{16}{27}, -\frac{32}{81}, \cdots$.
5. $-.3, .12, -.048, \cdots$.
6. $6, -3, \frac{3}{2}, \cdots$.

Find the values of the following:

7. $.4777 \cdots$. 8. $.8181 \cdots$. 9. $.5243243 \cdots$. 10. $.207575 \cdots$.

243. Geometric Means.

We define *inserting m geometric means between two numbers, a and b*, as finding a geometric progression of $m+2$ terms, whose first and last terms are a and b.

Ex. Insert 5 geometric means between 2 and $\frac{128}{729}$.

We find a geometric progression of 7 terms, in which $a = 2$, and $l = \frac{128}{729}$; substituting $n = 7$, $a = 2$, and $l = \frac{128}{729}$ in (I),

$$\frac{128}{729} = 2\,r^6\text{; whence } r^6 = \frac{64}{729}\text{, and } r = \pm\,\frac{2}{3}.$$

The result is $2, \pm\,\frac{4}{3}, \frac{8}{9}, \pm\,\frac{16}{27}, \frac{32}{81}, \pm\,\frac{64}{243}, \frac{128}{729}$.

244. Let x denote the geometric between a and b.

Then, $\qquad \dfrac{x}{a} = \dfrac{b}{x}$, or $x^2 = ab$.

Whence, $\qquad x = \sqrt{ab}$.

That is, *the geometric mean between two numbers is equal to the square root of their product.*

245. Problems.

1. The sixth term of an arithmetic progression is $\frac{5}{6}$, and the fifteenth term is $\frac{16}{3}$. Find the first term.

By § 230, the sixth term is $a + 5\,d$, and the fifteenth term $a + 14\,d$.

Then, by the conditions, $\begin{cases} a + 5\,d = \frac{5}{6}. & (1) \\ a + 14\,d = \frac{16}{3}. & (2) \end{cases}$

Subtracting (1) from (2), $\qquad 9\,d = \frac{9}{2}$; whence, $d = \frac{1}{2}$.

Substituting in (1), $\qquad a + \frac{5}{2} = \frac{5}{6}$; whence, $a = -\frac{5}{3}$.

2. Find four numbers in arithmetic progression such that the product of the first and fourth shall be 45, and the product of the second and third 77.

Let the numbers be $x-3y$, $x-y$, $x+y$, and $x+3y$.

Then by the conditions, $\begin{cases} x^2-9y^2=45. \\ x^2-\ \ y^2=77. \end{cases}$

Solving these equations, $x=9$, $y=\pm 2$; or, $x=-9$, $y=\pm 2$ (§ 224)
Then the numbers are 3, 7, 11, 15; or, -3, -7, -11, -15.

In problems like the above, it is convenient to represent the unknown numbers by *symmetrical* expressions.

Thus, if five numbers had been required, we should have represented them by $x-2y$, $x-y$, x, $x+y$, and $x+2y$.

3. Find 3 numbers in geometric progression such that their sum shall be 14, and the sum of their squares 84.

Let the numbers be represented by a, ar, and ar^2.

Then, by the conditions, $\begin{cases} a+ar+ar^2=14. & (1) \\ a^2+a^2r^2+a^2r^4=84. & (2) \end{cases}$

Divide (2) by (1), $a-ar+ar^2=6.$ (3)

Subtract (3) from (1), $2ar=8$, or $r=\dfrac{4}{a}\cdot$ (4)

Substituting in (1), $a+4+\dfrac{16}{a}=14$, or $a^2-10a+16=0$.

Solving this equation, $a=8$ or 2.

Substituting in (4), $r=\dfrac{4}{8}$ or $\dfrac{4}{2}=\dfrac{1}{2}$ or 2.

Then, the members are 2, 4, and 8.

EXERCISE 61

1. The seventh term of an A. P. is $\frac{64}{15}$, the twenty-first term is $\frac{38}{3}$. Find the fifteenth term.

2. Show that the sum of the odd integers from 1 to 999 is the square of their number.

3. The first term of an A. P. is 1, the sum of the third and ninth terms is 32. Find the sum of the first thirteen terms.

4. The sum of the first ten terms of an A. P. is to the sum of the first seven terms as 29 to 14. Find the ratio of the common difference to the first term.

5. There are four numbers, such that the first three form a G. P., the last three form an A. P. The sum of the first three is 73, of the last three 192. The difference between the second and fourth is 112. Find the numbers.

6. How many arithmetic means are inserted between $-\frac{3}{2}$ and $\frac{9}{2}$, when their sum is $\frac{21}{2}$?

7. Find four numbers in A. P., such that the sum of the first and second shall be -1, and the product of the second and fourth 24.

8. A traveller sets out from a certain place, and goes 7 miles the first hour, $7\frac{1}{2}$ the second hour, 8 the third hour, and so on. After he has been gone 5 hours, another sets out and travels $16\frac{1}{4}$ miles an hour. How many hours after the first starts are the travellers together?

9. If a person saves $120 each year, and puts the sum at simple interest at $3\frac{1}{2}\%$ at the end of each year, to how much will his property amount at the end of 18 years?

10. A ball is dropped from a window 32 feet above the pavement. Assuming the ball to be perfectly elastic and that on each rebound it rises to within $\frac{1}{6}$ of its former height, how far does it travel before coming to rest?

11. Two men travel from P to Q, leaving P at the same time. The distance from P to Q is 63 miles. The first travels 1 mile the first hour, 2 miles the second hour, 4 miles the third hour, and so on. The second travels 11 miles the first hour, $10\frac{4}{5}$ miles the second hour, $10\frac{3}{5}$ miles the third hour, and so on. Which is first to arrive at Q?

12. Find the geometric mean between .0729 and .0529.

13. Find the geometric mean between $\dfrac{.0144}{2.25}$ and $\dfrac{.0625}{576.}$.

14. Find the geometric mean between $\frac{x^2+xy}{xy-y^2}$ and $\frac{x^2-y^2}{xy}$.

15. Find the geometric mean between a^2-4a+4 and $4a^2+4a+1$.

16. The product of the first five terms of a G.P. is 243. Find the third term.

17. The digits of a number of three figures are in geometric progression. If units' and tens' digits are interchanged, the number formed exceeds the original number by 36. The sum of the digits is 14. Find the number.

18. A man travels $445\frac{1}{2}$ miles. He travels 10 miles the first day, and increases his speed one-half mile in each succeeding day. How many days does the journey require?

19. An A.P. has 19 terms such that the sum of the three middie terms is 3, and the sum of the first term and the last two terms is -13. Find the series.

20. Find the number of arithmetic means between 1 and 69, such that the ratio of the last mean to the first mean is 13.

21. Find an A.P. of 17 terms such that the sum of the first three terms is to the last term as 3 to 13, the first term being unity.

22. The sum of three successive terms of a geometric progression is 39 and the sum of their squares is 819. Find the series.

23. The sum of how many terms of the series 1, 3, 9 ⋯, is 3280?

24. Show that in any G.P., if each term is subtracted from the succeeding term, the differences form a G.P.

25. Find three numbers in A.P., such that the square of the first added to the product of the other two gives 16, and the square of the second added to the product of the other two gives 14.

Hindoo formula

$$x = \frac{-b \pm \sqrt{b^2+4ac}}{2a}$$

$$ax^2 + bx = c$$

Math